CONCERNING TWO CLASSES OF REMARKABLE PERFECT SQUARE PAIRS*

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1. Introduction. Open a table of integer squares. If you happen to have one which includes several systems of numeration, you will find a great variety of numbers of alluring forms and properties [1]. Thus, with base $B=7$, $4444=(55)^2$, $2424=(42)^2$, $664664004004=(665665)^2$; and with $B=10$, $7744=(88)^2$, $165165836836=(406406)^2$. We do not know any exercises more instructive or more pleasing than to seek those perfect squares in which the pattern of the digits naturally arouses the curiosity of the mind.

In investigating such squares one encounters at every step opportunities for solving Diaphantine equations and for applying many of the beautiful theorems of arithmetic. To discover new examples† of notable four-digit squares the present paper follows two distinct lines of investigation.

2. A system of numeration in which there exist simultaneously squares $abab=x^2$ and $baba=y^2$. There is an infinity of systems of numeration in which there exist four-digit squares of the form [2]

$$M = x^2 = abab = (ab)(B^2 + 1). \tag{1}$$

Since $(ab)<x<B^2+1$, evidently B^2+1 must contain a square factor. Hence we put

$$B^2 + 1 = nm^2, \tag{2}$$

$$ab = aB + b = np^2, \tag{3}$$

with $m>p$. Moreover, since m and n divide B^2+1, each is itself a sum of two squares [3]. When the integers m, n, p exist, M is the square of the product $mnp=x$. In the particular case $n=2$, the relation (2) reduces to Fermat's equation

$$B^2 - 2m^2 = -1, \tag{4}$$

and we have $M=(2pm)^2$. The values of B are

$$B = 7, 41, 239, 1393, 8119, \cdots, \tag{4'}$$

for which the corresponding values of m are

$$m = 5, 29, 169, 985, 5741, \cdots. \tag{4''}$$

It will be seen that the values of B, m are the N_{2p+1}, D_{2p+1} as given below in Section 3.

In a system of numeration in which there exist simultaneously four-digit

* Translated from the French by R. E. Luce.

† We have previously pointed out a good many of these, Les Récréations Mathématiques, Supplément a Mathesis, 1943, pp. 24–88.

squares x^2 and y^2, where

$$x^2 = abab = (ab)(B^2+1), \qquad y^2 = baba = (ba)(B^2+1),$$

we must have

(5) $$B^2+1 = nm^2,\ ab = np^2,\ ba = nq^2,\ x^2 = (mnp)^2,\ y = (mnq)^2.$$

Since $ab,\ ba < B^2$, the process of obtaining squares having these forms will be a matter, first, of determining systems of numeration in which there shall be, at the same time, two-digit numbers of form np^2 and nq^2 such that the sum

(6) $$ab + ba = n(p^2+q^2) = (a+b)(B+1).$$

Thus, when n is a sum of two squares, the same is true of $n(p^2+q^2)$ and hence each of the numbers $a+b$ and $B+1$ can be decomposed into a product of prime factors, each of which is, in its turn, a sum of two squares.

Consider further the particular case noted above, with $n=2=1^2+1^2$. In the sequence (4′) of values of B, it is necessary to exclude $B=41, 239, 8119, \cdots$ for which $B+1$ is divisible by $3, 7, \cdots$, for primes of the form $4k-1$ are not sums of two squares. But the numbers $B=7, 1393, \cdots$, for which $7+1=2^2+2^2$, $1393+1=13^2+35^2, \cdots$, fulfill the two conditions (4) and (6) for B^2+1 and $B+1$. It will suffice therefore to discover whether there exist in these systems of numeration two-digit numbers ab, ba, each of which, according to (5) is the double of a square. For $B=7$, we find

$$ab = 2\cdot 4^2 = 44 = ba, \qquad x^2 = 4444 = (55)^2 = y^2.$$

For $B=1393$, there are several examples:

(i) $$x^2 = 2\ \overline{912}\ 2\ \overline{912} = (\overline{60}\ \overline{1130})^2, \qquad y^2 = \overline{912}\ 2\ \overline{912}\ 2 = (\overline{1127}\ \overline{179})^2.$$

(ii) $$x^2 = 2\ \overline{46}\ 4\ \overline{46} = (\overline{74}\ \overline{1328})^2, \qquad y^2 = \overline{46}\ 4\ \overline{46}\ 4 = (\overline{253}\ \overline{201})^2.$$

The numbers $(2x)^2$, $(3x)^2$, $(4x)^2$ are equally suitable.

(iii) $$x^2 = 4\ \overline{478}\ 4\ \overline{478} = (\overline{77}\ \overline{1089})^2, \qquad y^2 = \overline{478}\ 4\ \overline{478}\ 4 = (\overline{816}\ 2)^2.$$

Doubtless other examples exist which may be discovered by extended study of the further values of B in (4′) and from values of B found by using $n=5, 10, 13, 17, \cdots$ with the resulting equation analogous to (4).

3. Perfect squares $aabb=(cc)^2$ and $bbaa=(dd)^2$ in a single system of numeration. For squares having the prescribed forms it is easily shown [4] that the base B and the digits a, b, c, d necessarily satisfy the relations

(1) $$a+b = B+1,$$

(2) $$c^2 = a(B-1)+1, \qquad d^2 = b(B-1)+1,$$

(3) $$c^2+d^2 = B^2+1,$$

(4) $$c^2-d^2 = (a-b)(B-1).$$

Moreover if B is odd, c and d are necessarily odd, and a and b are of like parity, so that one can always put $a-b=2s$. (We take $s \geqq 0$, that is $a \geqq b$. If $a<b$ we merely interchange a with b, and c with d.) By virtue of the definition of s and equation (1), the relations (2) are equivalent to

$$(B+s)^2 - 2c^2 = (s+1)^2 - 2, \tag{5}$$

$$(B-s)^2 - 2d^2 = (s-1)^2 - 2. \tag{6}$$

A general formula yielding solutions may be obtained* from the familiar solution of Fermat's equations

$$x^2 - 2y^2 = \pm 1$$

in which are employed successive approximants of the continued fraction representation of $\sqrt{2}$. If N_k/D_k is the kth approximant, we have

k	1	2	3	4	5	6	7	8	9	$10\cdots$
N_k	1	3	7	17	41	99	239	577	1393	$3363\cdots$
D_k	1	2	5	12	29	70	169	408	985	$2378\cdots$

For this development the following relations are known:

$$N_{k+1} = N_k + 2D_k, \qquad D_{k+1} = N_k + D_k,$$

$$2N_k = N_{k+1} - N_{k-1}, \qquad 2D_k = D_{k+1} - D_{k-1},$$

$$N_{2p}^2 - 2D_{2p}^2 = 1, \qquad N_{2p-1}^2 - 2D_{2p-1}^2 = -1,$$

$$N_{2p} \cdot N_{2p-1} - 2D_{2p} \cdot D_{2p-1} = -1, \qquad N_{2p+1} \cdot N_{2p} - 2D_{2p+1} \cdot D_{2p} = 1.$$

From these equations, the following identities in s are easy to obtain:

$$(N_{2p}s + N_{2p-1})^2 - 2(D_{2p}s + D_{2p-1})^2 = (s-1)^2 - 2,$$

$$(N_{2p}s - N_{2p-1})^2 - 2(D_{2p}s - D_{2p-1})^2 = (s+1)^2 - 2,$$

$$(N_{2p}s + N_{2p+1})^2 - 2(D_{2p}s + D_{2p+1})^2 = (s+1)^2 - 2,$$

$$(N_{2p}s - N_{2p+1})^2 - 2(D_{2p}s - D_{2p+1})^2 = (s-1)^2 - 2.$$

These results suggest solutions of (5) and (6) obtained by putting

$$B + s = N_{2p}s - N_{2p-1}, \qquad c = D_{2p}s - D_{2p-1};$$

$$B - s = N_{2p}s - N_{2p+1}, \qquad d = D_{2p}s - D_{2p+1};$$

$$B' + s = N_{2p'}s + N_{2p'+1}, \qquad c' = D_{2p'}s + D_{2p'+1};$$

$$B' - s = N_{2p'}s + N_{2p'-1}, \qquad d' = D_{2p'}s + D_{2p'-1}.$$

We may now take $p=p'$ and equate the values of B and B' corresponding to c and d, c' and d', c' and d, thus obtaining

$$B_1 = (N_{2p} - 1)s - N_{2p-1} = (N_{2p} + 1)s - N_{2p+1},$$

* According to notes communicated to the author by M. A. Barriol, Paris.

$$B_2 = (N_{2p} - 1)s + N_{2p+1} = (N_{2p} + 1)s + N_{2p-1},$$
$$B_3 = (N_{2p} - 1)s + N_{2p+1} = (N_{2p} + 1)s - N_{2p+1}.$$

(The fourth combination, c and d', gives s a negative value.) For B_1 and B_2 we find $2s = N_{2p+1} - N_{2p-} = 2N_{2p}$; and for B_3, $s = N_{2p+1}$. We have two solutions therefore for each $s = N_{2p}$ and one for each $s = N_{2p+1}$.

We have thus three infinite sequences of solutions:

(i) $s = N_{2p}$, $B = (N_{2p} - 1)N_{2p} - N_{2p-1}$,
$c = D_{2p}N_{2p} - D_{2p-1}$, $d = D_{2p}N_{2p} - D_{2p+1}$;

(ii) $s = N_{2p}$, $B = (N_{2p} + 1)N_{2p} + N_{2p-1}$,
$c = D_{2p}N_{2p} + D_{2p+1}$, $d = D_{2p}N_{2p} + D_{2p-1}$;

(iii) $s = N_{2p+1}$, $B = N_{2p}N_{2p+1}$,
$c = D_{2p}N_{2p+1} + D_{2p+1}$, $d = D_{2p}N_{2p+1} - D_{2p+1}$.

The values of a and b are found from $a-b=2s$, $a+b=B+1$. The following table indicates the simplest of these solutions:

s	B	c	d	a	b
3	13	11	7	10	4
7	21	19	9	18	4
17	265	199	175	150	116
17	313	233	209	174	140
41	697	521	463	390	308

Complete solutions of (5) and (6) present genuine difficulties, but there is another approach which leads easily to additional sequences of squares of the desired forms.

The general solution in integers of (3) is known [5] to be

(7) $$B = mq + np, \quad c = mp + nq, \quad d = |mq - np|,$$

subject to the condition

(8) $$mp - nq = 1.$$

Equations (2) may now be solved in terms of m, n, p, q, giving

(9) $$a = \frac{4mnpq}{n(p+q) - m(p-q)},$$

(10) $$b = \frac{(m-n)(m+n)(q-p)(q+p)}{n(p+q) - m(p-q)}.$$

The necessary and sufficient conditions that a and b be integers is therefore

simply that $n(p+q)-m(p-q)$ be a divisor of $4mnpq$. If in addition

$$0 < a < B, \quad 0 < b < B, \quad 0 < c < B, \quad 0 < d < B, \tag{11}$$

then the digits a, b, c, d determine perfect squares of the type under discussion, the base being B.

It is possible to assume additional relations between m, n, p, q in such a way that the conditions required for a in (9) to be integral are easily fulfilled. For example if $n=p-q$, then (9) reduces to $a=4mpq/(p+q-m)$. If, further, $m=p-2q$, this becomes simply $a=4mp/3$. With these values of n and m, condition (7) becomes $p^2-3pq+q^2=1$ or

$$(2p - 3q)^2 - 5q^2 = 4.$$

Since this last equation is known to have infinitely many integral solutions one obtains an unending sequence of pairs of perfect squares of the desired form. The values of B are 7, 313, 14686, $\cdots$.

Other conditions on m and n lead similarly to other sequences of satisfactory squares. The following are easy to determine:

$$\begin{aligned}
&n = p - q, m = p - 3q; \quad B = 13, 177, 2461, \cdots . \\
&m = p + q, \ n = 2q - 2p; \ B = 526, \cdots . \\
&m = p + q, \ n = p + 2q; \quad B = 31, 35491, \cdots . \\
&m = p + q, \ n = p + 3q; \quad B = 13, 177, 2461, \cdots . \\
&m = p + q, \ n = p + 5q; \quad B = 313, \cdots . \\
&m = p + q, \ n = 5p - q; \quad B = 305, 4261, \cdots . \\
&m = p + q, \ n = 4p - q; \quad B = 265, 12541, \cdots .
\end{aligned}$$

The following particular cases should be noted. (i) With $a=b$ so that

$$aaaa = bbbb = (cc)^2 = (dd)^2,$$

equation (6) becomes $B^2-2c = -1$, and the solutions are immediate:

$$B = N_{2p+1}, \qquad c = d = D_{2p+1}, \qquad a = b = (B + 1)/2.$$

If we put (ii) $b=4$, as we have already shown [2], we have, for any base $B=n^2+n+1$,

$$\begin{aligned}
&\overline{B-3}\ \overline{B-3}\ 4\ 4 = (\overline{B-2}\ \overline{B-2})^2, \\
&4\ 4\ \overline{B-3}\ \overline{B-3} = (\overline{2n+1}\ \overline{2n+1})^2.
\end{aligned}$$

Selected examples are shown in the following table. The special cases just mentioned are excluded. As noted earlier, interchange of a and b, and of c and d does not give an essentially different solution. Although it does introduce a new set of values of m, n, p, q, such a second representation is not included.

m	n	p	q	B	a	b	c	d
4	1	5	19	81	19	63	39	71
11	19	7	4	177	133	45	153	89
19	5	4	15	305	75	231	151	265
5	13	21	8	313	140	174	209	233
25	14	9	16	526	384	143	449	274
15	4	19	71	1141	284	858	569	989
71	19	15	56	4261	1064	3198	2129	3691
34	89	144	55	14686	6528	8159	9791	10946

References

1. Thébault, Les Récréations Mathématiques (Parme les nombres curieux), Supplément a Mathesis, 1943, pp. 88–120.
2. V. Thébault, Mathesis, 1933, p. 385.
3. Kraitchik, Theorie des Nombres, 1922, p. 98.
4. V. Thébault, Mathesis, 1936, Supplément, p. 5. See also this MONTHLY, problem 4237 [1948, 370].
5. L' Intermédiare des Mathématiciens, 1910, pp. 96, 118, 204.
6. Thébault, Ann. de la Soc. Scient. de Bruxelles, t. lxii, 1948, pp. 101–105.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the ninth annual William Lowell Putnam Mathematical Competition held March 26, 1949, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, and is held under the auspices of The Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were W. F. Stinespring, D. L. Yarmush, Ariel Zemach; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario, Canada. The members of the team were J. M. Kennedy, J. E. S. Moyse, J. B. Patterson; to each of these a prize of thirty dollars is awarded.

* Director of the Competition.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Carnegie Institute of Technology, Pittsburgh, Pennsylvania. The members of the team were G. L. Baldwin, R. E. Cutkosky, R. M. Drisko; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the College of the City of New York, New York, New York. The members of the team were Herman Hanisch, D. J. Newman, H. R. Shapiro; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were J. W. Milnor, Princeton University; D. J. Newman, College of the City of New York; W. F. Stinespring, Harvard University; D. L. Yarmush, Harvard University; Ariel Zemach, Harvard University. Each of these will receive a prize of fifty dollars.

The four succeeding persons ranking highest in the examination, named in alphabetical order, were: J. M. Kennedy, University of Toronto; J. D. Lordan, Massachusetts Institute of Technology; Gerhard Raynar, Harvard University; W. F. Reynolds, Holy Cross College. To each of these a prize of twenty dollars is awarded.

There was a tie for tenth place between R. E. Cutkosky, Carnegie Institute of Technology, and Louis Howard, Swarthmore College. A prize of twenty dollars will be divided between these contestants.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being H. A. Forrester, H. G. Hegnemann, J. A. Hummel; Polytechnic Institute of Brooklyn, Brooklyn, New York, the members of the team being F. Blecher, George Giovmousis, S. Lerner; Princeton University, Princeton, New Jersey, the members of the team being C. H. Bernstein, P. H. Lord, J. W. Milnor; Swarthmore College, Swarthmore, Pennsylvania, the members of the team being Louis Howard, Paul Mangelsdorf, and Robert Norman.

Five individuals were given honorable mention. The names are listed in alphabetical order. R. M. Drisko, Carnegie Institute of Technology; E. O. Elliott, University of California at Berkeley; M. L. Minsky, Harvard University; J. E. S. Moyse, University of Toronto; H. R. Shapiro, College of the City of New York.

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: American International College, Brooklyn College, California Institute of Technology, Carnegie Institute of Technology, College of the City of New York, College of St. Thomas, Columbia University, George Pepperdine College, Harvard University, Holy Cross College, Macalester College, McGill University, McMaster University, Northwest Missouri State Teachers College, Oklahoma A. & M. College, Polytechnic Institute of Brooklyn, Princeton University, Queen's University, Reed College, Saint Joseph's College (Philadelphia), Stanford University, Swarthmore College, United States Naval Academy, University of Alberta, University

of British Columbia, University of Buffalo, University of California at Berkeley, University of California at Los Angeles, University of New Hampshire, University of Oregon, University of Toronto, Ursinus College, and Washburn Municipal University.

The following additional colleges and universities entered individual contestants only: Carleton College, Duquesne University, Haverford College, Hofstra College, Massachusetts Institute of Technology, Occidental College, Purdue University, Radford College, St. Joseph College (West Hartford), St. Xavier College, University of Minnesota, University of Omaha, University of Pittsburgh, University of Saskatchewan, University of Wyoming, and Wiley College.

A total of 155 undergraduates representing 49 institutions took part in the Competition.

Participants in the Competition were given the following lists of problems.

PART I. THREE HOURS

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. Answer *either* (a) or (b):

   (a) Three straight lines pass through the three points $(0, -a, a)$, $(a, 0, -a)$, and $(-a, a, 0)$, parallel to the x-axis, y-axis, and z-axis, respectively; $a>0$. A variable straight line moves so that it has one point in common with each of the three given straight lines. Find the equation of the surface described by the variable line.

   (b) Which planes cut the surface $xy+xz+yz=0$ in (*i*) circles, (*ii*) parabolas?

2. We consider three vectors drawn from the same initial point O, of lengths a, b, and c, respectively. Let E be the parallelepiped with vertex O of which the given vectors are the edges and H the parallelepiped with vertex O of which these vectors are the altitudes. Show that the product of the volumes of E and H equals $(abc)^2$, and generalize the theorem, with proof, to n dimensions.

3. Assume that the complex numbers $a_1, a_2, \cdots, a_n, \cdots$ are all different from zero, and that $|a_r-a_s|>1$ for $r\neq s$. Show that the series

$$\sum_{n=1} \frac{1}{a_n^2}$$

   converges.

4. Given that P is a point inside a tetrahedron with vertices at A, B, C, and D, such that the sum of the distances $PA+PB+PC+PD$ is a minimum. Show

that the two angles $\angle APB$ and $\angle CPD$ are equal, and are bisected by the same straight line. What other pairs of angles must be equal?

5. How many roots of the equation $z^6+6z+10=0$ lie in each quadrant of the complex plane?

6. Prove that for every real or complex x

$$\prod_{k=1}^{\infty} \frac{1+2\cos\frac{2x}{3^k}}{3} = \frac{\sin x}{x}.$$

PART II. THREE HOURS

(Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.)

1. Each rational number p/q (p, q relatively prime positive integers) of the open interval (0, 1) is covered by a closed interval of length $1/2q^2$, whose center is at p/q. Prove that $\sqrt{2}/2$ is not covered by any of the above closed intervals.

2. Answer either (a) or (b):
   (a) Prove that

$$\sum_{n=2}^{\infty} \frac{\cos(\log\log n)}{\log n}$$

   diverges.
   (b) Assume that $p>0$, $a>0$, and $ac-b^2>0$, and show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dxdy}{(p+ax^2+2bxy+cy^2)^2} = \pi p^{-1}(ac-b^2)^{-1/2}.$$

3. Let K be a closed plane curve such that the distance between any two points of K is always less than 1. Show that K lies inside a circle of radius $1/\sqrt{3}$.

4. Show that the coefficients $a_1, a_2, a_3, \cdots$ in the expansion $\frac{1}{4}[1+x-(1-6x+x^2)^{1/2}]=a_1x+a_2x^2+a_3x^3+\cdots$ are positive integers.

5. Let $a_1, a_2, \cdots, a_n, \cdots$ be an arbitrary sequence of positive numbers. Show that

$$\limsup_{n\to\infty}\left(\frac{a_1+a_{n+1}}{a_n}\right)^n \geqq e.$$

6. Let C be a closed convex curve with a continuously turning tangent and let O be a point inside C. With each point P on C we associate the point $T(P)$ on

C which is defined as follows: Draw the tangent to C at P and from O drop the perpendicular to that tangent. $T(P)$ is then the point at which this perpendicular intersects the curve C.

Starting now with a point P_0 on C, we define points P_n by the formula $P_n = T(P_{n-1})$, $n \geqq 1$. Prove that the points P_n approach a limit, and characterize those points which can be limits of sequences P_n. (You may consider the facts that T is a continuous transformation and that a convex curve lies on one side of each of its tangents as not requiring proofs.)

ON GENERALIZED CAUCHY FUNCTIONAL EQUATIONS

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1. Introduction. In a previous paper* use was made of Cauchy's functional equation $f(x+y) = f(x)f(y)$ to obtain a generalization of trigonometry. In the present paper much more general functional equations are studied, and it is shown that the solutions of one of these equations can be made the basis for the definition of certain functions which have many properties analogous to those of the ordinary trigonometric functions.

2. The basic functional equations. We consider the functional equation

$$(A) \qquad F(x + y + nxy) = G(x)H(y), \qquad (n > 0)$$

and set ourselves the problem of finding the most general real, continuous not identically vanishing functions $F(x)$, $G(x)$ and $H(x)$ which are defined for all x greater than $-1/n$ and satisfy this equation (A). A solution which consists of functions which do not vanish identically will be called a non-trivial solution. We first state and prove

THEOREM I. *Every non-trivial solution of* (A) *is of the form* $G(x) = G(0)f_n(x)$, $H(x) = H(0)f_n(x)$, $F(x) = G(0)H(0)f_n(x)$ *where* $f_n(x)$ *is any non-trivial solution*† *of*

$$(B) \qquad f_n(x + y + nxy) = f_n(x)f_n(y), \qquad (n > 0)$$

defined for all x *greater than* $-1/n$, *where* $G(0) \neq 0$, $H(0) \neq 0$.

Proof. If in (A) we let first $y = 0$, and then $x = 0$, we obtain the equations

$$(2.1) \qquad F(x) = G(x)H(0) \qquad F(y) = G(0)H(y).$$

* Thielman, H. P. A generalization of trigonometry. Nat. Math. Mag., Vol. XI (1937) pp. 1–3.

† Cauchy treated this equation with $n = 0$, and related equations. Cours d'Analyse (1821), Chapter 5. It is because of his study of these equations that his name is attached to them.

If $H(0)=0$, or $G(0)=0$, then $F(x)\equiv 0$. Since we are interested in the non-trivial solutions, $H(0)$ and $G(0)$ must be assumed to be different from zero. From (2.1) it follows that $F(0)=G(0)H(0)$, and $G(x)=F(x)/H(0)$, $H(x)=F(x)/G(0)$. Substituting these values for $G(x)$ and $H(x)$ in (A) we get

$$F(x+y+nxy)=F(x)F(y)/G(0)H(0),$$

which can be written as

$$\frac{F(x+y+nxy)}{G(0)H(0)}=\frac{F(x)}{G(0)H(0)}\cdot\frac{F(y)}{G(0)H(0)}. \tag{2.2}$$

If we let $f_n(x)=F(x)/G(0)H(0)$, we see from (2.2) that $f_n(x)$ satisfies equation (B), and since $G(x)=G(0)f_n(x)$, $H(x)=H(0)f_n(x)$, $F(x)=G(0)H(0)f_n(x)$, the theorem is proved.

The subscript n on $f_n(x)$ in equation (B) is used to indicate that this function may depend on the parameter n which occurs in the functional equation. If $n=0$ in the functional equation (B), it reduces to Cauchy's functional equation. We shall show that the limit as n goes to zero of certain continuous solutions of (B) will reduce to the continuous solutions of Cauchy's equation, that is, to A^x where A is any positive constant.

Theorem I shows that the problem of solving equation (A) can be reduced to that of solving the simpler equation (B). We proceed now to the study of this latter equation.

For equation (B) we have the

THEOREM II. *Every non-trivial, real continuous solution $f_n(x)$ of the functional equation (B) which is defined for all $x>-1/n$, is of the form $f_n(x)=(1+nx)^k$, where k is any real number.*

Proof. We note that (B) can be written as

$$f_n\left[\frac{(1+nx)(1+ny)-1}{n}\right]=f_n(x)f_n(y). \tag{2.3}$$

Let $1+nx=e^u$, $1+ny=e^v$, then the last displayed equation may be written as

$$f_n\left[\frac{e^{u+v}-1}{n}\right]=f_n\left(\frac{e^u-1}{n}\right)f_n\left(\frac{e^v-1}{n}\right).$$

Let $F(u)=f_n[(e^u-1)/n]$. Then because of the last displayed equation we get

$$F(u+v)=F(u)F(v), \tag{2.4}$$

which is Cauchy's functional equation. It is however well known that all real continuous solutions of this equation are of the form $F(u)=e^{ku}$ where k is any real number. Hence all continuous solutions of (B) are of the form $f_n[(e^x-1)/n]=e^{kx}$, or $f(t)=(1+nt)^k$.

It might be of interest to show that the limit as n goes to zero of certain

solutions of (B), that is, those for which $k=c/n$, give the non-trivial solutions of Cauchy's functional equation as it was mentioned earlier in the paper. To this end we write

$$\lim_{n\to 0} f_n(t) = \lim_{n\to 0} (1+nt)^{c/n} = \lim_{n\to 0} [(1+nt)^{1/nt}]^{ct} = e^{ct}.$$

THEOREM III. *Every real, non-trivial, continuous solution of the functional equation (A) which is defined for all $x>-1/n$ is of the form*

$$G(x) = G(0)(1+nx)^k, \qquad H(x) = H(0)(1+nx)^k,$$

$F(x)=G(0)H(0)(1+nx)^k$, where k is any real number.

Proof. This result is a direct consequence of Theorems I and II.

THEOREM IV. *All non-trivial, real, continuous solutions of the functional equation*

$$(C) \qquad g_n(x+y+nxy) = g_n(x) + g_n(y), \qquad n>0,$$

which are defined for all $x>-1/n$, are of the form $g_n(x)=k\log(1+nx)$, where k is any real number.

Proof. Let $f_n(x)=e^{g_n(x)}$. Then $f_n(x)$ is a non-trivial continuous solution of (B). Hence $g_n(x)=k\log(1+nx)$.

From Theorem III we obtain

THEOREM V. *All non-trivial, real, continuous solutions of the functional equation*

$$f(x+y+nxy) = g(x)+h(y) \qquad n>0$$

are of the form

$$g(x) = k\log(1+nx)+g(0)$$
$$h(x) = k\log(1+nx)+h(0)$$
$$f(x) = k\log(1+nx)+g(0)+h(0)$$

where k is any real number.

Proof. Let $F(x)=e^{f(x)}$, $G(x)=e^{g(x)}$, $H(x)=e^{h(x)}$. Then $F(x+y+nxy)=G(x)\cdot H(y)$, and the result follows from Theorem III.

Many interesting results‡ obtained on discontinuous solutions of the equations (2.4) and (C) (with $n=0$) can also be extended by means of transformations of variables to the equations (B) and (C) of this paper.

3. Functions which are analogous to trigonometric functions. Let $f_n(x)$ be any non-trivial solution of the functional equation (B). We know that $f_n(0)=1$.

‡ Hamel, Mathematische Annalen, Vol. LX (1905), p. 459; Ostrowski, Comm. Math. Helvetici, 1 (1929) pp. 157–159.

We define two functions $s_n(x)$ and $c_n(x)$ by the equations

$$(3.1)\quad s_n(x) = \frac{f_n(x) - f_n\left(\frac{-x}{1+nx}\right)}{2h}, \qquad c_n(x) = \frac{f_n(x) + f_n\left(\frac{-x}{1+nx}\right)}{2}$$

where $h \neq 0$ is a constant. We see that $s_n(0)=0$, $c_n(0)=1$, $s_n[(-x/1)+nx] = -s_n(x)$, $c_n(-x/1+nx)=c_n(x)$. Multiplying the first equation of (3.1) first by h, and then by $-h$, and adding each of the results to the second one of equations (3.1), we obtain

$$(3.2)\quad \begin{aligned} c_n(x) + hs_n(x) &= f_n(x), \\ c_n(x) - hs_n(x) &= f_n\left(\frac{-x}{1+nx}\right). \end{aligned}$$

Multiplying the last two equations we get, because of (B), Pythagoras' theorem,

$$(3.3)\quad c_n^2(x) - h^2 s_n^2(x) = 1.$$

It is rather interesting to note that if we let $f_n(x)$ be the particular continuous solution of (B) $(1+nx)^{h/n}$, then (3.3) shows that Pythagoras' theorem is satisfied by a one parameter family of simple algebraic functions

$$s_n(x) = \frac{(1+nx)^{h/n} - (1+nx)^{-h/n}}{2h}, \qquad c_n(x) = \frac{(1+nx)^{h/n} + (1+nx)^{-h/n}}{2},$$

which, with $h=1$, or $h=i$, continuously approach the hyperbolic, or circular functions as n goes to zero.

We next note that any solution $f_n(x)$ of (B) satisfies the functional equation

$$(3.4)\quad f\left[\frac{(1+nx)^r - 1}{n}\right] = f^r(x),$$

where r is any rational number. This result could be derived directly from (B), but it is quicker to derive it from Cauchy's equation (2.4). Let $F(x)$ be any solution of (2.4). Then $F(rx)=F^r(x)$, where r is any rational number, which is a well known result. But if $f_n(x)$ is any solution of (B), then $F(x)=f_n(e^x-1/n)$ is a solution of (2.4), and hence

$$f_n\left(\frac{e^{rx}-1}{n}\right) = f_{nr}\left(\frac{e^x-1}{n}\right).$$

Setting $e^x = 1+nx$, the last displayed equation reduces to (3.4). If we assume that $f_n(x)$ is continuous, the functional equation (3.4), holds for all real values of r.

Raising each side of the first one of the equations (3.2) to the pth power, we get

$$[c_n(x) + hs_n(x)]^p = f_n^p(x) = f_n\left[\frac{(1+nx)^p - 1}{n}\right] \tag{3.5}$$

by (3.4). If $f_n(x)$ is assumed to be continuous this equation is true for all real values of p, otherwise it is true for all rational values only. Equation (3.5) can be written because of (3.2) as

$$[c_n(x) + hs_n(x)]^p = c_n\left[\frac{(1+nx)^p - 1}{n}\right] + hs_n\left[\frac{(1+nx)^p - 1}{n}\right], \tag{3.6}$$

which corresponds to De Moivre's theorem. Also by (3.2) and (*B*) we have

$$\begin{aligned} c_n(x+y+nxy) + hs_n(x+y+nxy) &= [c_n(x) + hs_n(x)][c_n(y) + hs_n(y)] \\ &= c_n(x)c_n(y) + hc_n(x)s_n(y) \\ &\quad + hc_n(y)s_n(x) + h^2 s_n(x)s_n(y), \end{aligned} \tag{3.7}$$

$$\begin{aligned} c_n(x+y+nxy) - hs_n(x+y+nxy) &= f_n\left[\frac{-(x+y+nxy)}{1+n(x+y+nxy)}\right] \\ &= f_n\left[\frac{-x}{1+nx}\right] f_n\left[\frac{-y}{1+ny}\right] \\ &= [c_n(x) - hs_n(x)][c_n(y) - hs_n(y)] \\ &= c_n(x)c_n(y) - hc_n(x)s_n(y) \\ &\quad - hc_n(y)s_n(x) + h^2 s_n(x)s_n(y). \end{aligned} \tag{3.8}$$

Adding and subtracting the equations (3.7) and (3.8) we get the addition formulas

$$\begin{aligned} c_n(x+y+nxy) &= c_n(x)c_n(y) + h^2 s_n(x)s_n(y) \\ s_n(x+y+nxy) &= s_n(x)c_n(y) + s_n(y)c_n(x). \end{aligned} \tag{3.9}$$

We note the similarity of these formulas to those of the cosine and sine functions. If we should use additional properties of the function $f_n(x)$ such as differentiability, we could derive many more properties which would have their analogues in ordinary trigonometry, or in the theory of hyperbolic functions.

The functions $s_n(x)$ and $c_n(x)$ could have been defined in terms of the ordinary hyperbolic functions of arguments which satisfy a certain functional equation. We might have written $s_n(x) = 1/h \sinh g_n(x)$, $c_n(x) = \cosh g_n(x)$ where $g_n(x)$ is any solution of the functional equation (*C*). This shows that the functions $s_n(x)$, $c_n(x)$ are particular cases of $s(x) = 1/h \sinh g(x)$, $c(x) = \cosh g(x)$ where $g(x)$ is any solution of $g(x) + g(y) = g[\mu(x, y)]$. Then it follows that

$$\begin{aligned} c[\mu(x, y)] &= c(x)c(y) + h^2 s(x)s(y), \\ s[\mu(x, y)] &= s(x)c(y) + c(y)s(y). \end{aligned}$$

In particular if $\mu(x, y) = x + y + nxy$, $s(x) = s_n(x)$, $c(x) = c_n(y)$. It is rather inter-

esting to note that if in this case $g(x)$ is assumed to be continuous then, by Theorem IV, $g(x)$ has to be $k \log (1+nx)$. The object of this section has been to derive a theory analogous to that of the hyperbolic and circular functions on the basis of a given functional equation. If one were interested in generalizing the results of this section one could start with an essentially different functional equation, or one could consider elements, and operations of some group or linear space, satisfying analogous functional equations. One could for example write $A(x+y+nxy)=A(x)\cdot A(y)$ where $A(x)$ might be a matrix and the operation $(\cdot)$ indicate matrix multiplication. Other generalizations that could be taken up here will suggest themselves to the reader.

ARITHMETICAL PROPERTIES OF SUMS OF POWERS

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1. Introduction. For each polynomial,

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n = \prod_{\nu=1}^{n} (x - x_\nu),$$

we define $s_q(p)=x_1^q+ \cdots +x_n^q$ $(q=1, 2, 3, \cdots)$. It is known that if all the a_i's are integers, then so are all the s_i's. We will say that an ordered set of integers,

$$S(1), S(2), \cdots, S(n),$$

has the property P provided there exists a polynomial p, having integral coefficients, for which $S(j)=s_j(p)$ $(i=1, 2, \cdots, n)$.

The following criterion concerning the sums s_i is due to Jänichen [1]:

The set S has the property P if and only if the congruences

$$\sum_{d|m} \mu(d)S(m/d) \equiv 0 \pmod{m} \qquad (m = 1, 2, \cdots, n)$$

all hold.

2. A generalization. The purpose of the first part of this note is to prove the following generalization of Jänichen's criterion.

THEOREM 1. *The set S has the property P if and only if the congruences*

$$\sum_{d|m} f(d)S(m/d) \equiv 0 \pmod{m} \qquad (m = 1, 2, \cdots, n) \tag{1}$$

all hold, where f is an arbitrary integer-valued function satisfying the conditions

$$f(1) = \pm 1, \tag{2}$$

$$\sum_{d|m} f(d) \equiv 0 \pmod{m} \qquad (m = 1, 2, \cdots, n). \tag{3}$$

In particular, f may be an arbitrary multiplicative function that satisfies (3) whenever m is a power of a prime, such as Möbius' function μ and Euler's function ϕ [2]. The example $f(1)=1$, $\sum_{d|m} f(d) = -m$ for $m>1$, whence $f(2)=-3$, $f(3)=-4$, $f(6)=0$, shows that the conditions (2), (3) do not imply that f is multiplicative.

COROLLARY. *If f is an arbitrary integer-valued function satisfying the conditions*

$$\sum_{d|m} f(d) \equiv 0 \pmod{m} \qquad \text{for each } m \geqq 1, \tag{3'}$$

then, for sums of powers $s_q(p)$,

$$\sum_{d|m} f(d)s_{m/d} \equiv 0 \pmod{m} \qquad \text{for each } m \geqq 1. \tag{1'}$$

In order to prove Theorem 1 and the corollary we need the three lemmas that follow. All the functions involved in these lemmas are defined for $m=1, 2, \cdots n$, where n is an arbitrary positive integer.

LEMMA 1. *If $f(m)$, $g(m)$, $R(m)$ are three functions satisfying the conditions*

$$g(1)R(1) = \pm 1, \tag{4}$$

$$\sum_{d|m} g(d)R(m/d) \equiv 0 \pmod{m}, \tag{5}$$

$$\sum_{d|m} f(d)R(m/d) \equiv 0 \pmod{m}, \tag{6}$$

then also

$$F(m) = \sum_{d|m} f(d)g'(m/d) \equiv 0 \pmod{m}, \tag{7}$$

where g' is the Dirichlet reciprocal of g, that is, $\sum_{d|m} g(d)g'(m/d) = 1$, 0 according as $m=1$ or $m>1$.

Proof. Relation (7) is evidently true for $m=1$. Suppose (7) is true for every divisor $d<m$ of m. Statement (7) implies $f(k) = \sum_{d|k} F(d)g(k/d)$. Hence

$$\begin{aligned}
0 \equiv \sum_{d|m} f(d)R(m/d) &= \sum_{d|m} \sum_{d'|d} F(d')g(d/d')R(m/d) = \sum_{d|m} \sum_{d'|d} F(m/d)g(d/d')R(d') \\
&= \sum_{d|m} F(m/d) \sum_{d'|d} g(d/d')R(d') \\
&= F(m)g(1)R(1) + \sum_{d|m,\, d<m} F(m/d) \sum_{d'|d} g(d/d')R(d') \\
&= \pm F(m) + \sum_{d|m,\, d<m} (m/d)Q(d)\cdot dQ'(d) \equiv \pm F(m) \pmod{m}
\end{aligned}$$

(Q, Q' integer-valued).

LEMMA 2. *If $f(m)$, $g(m)$, $R(m)$, $S(m)$ are four functions satisfying the conditions* (4), (5), (6) *and*

$$\sum_{d|m} g(d)S(m/d) \equiv 0 \pmod{m}, \tag{8}$$

then also

$$\sum_{d|m} f(d)S(m/d) \equiv 0 \pmod{m}. \tag{7'}$$

Proof. By Lemma 1, we have

$$\sum_{d|m} f(d)S(m/d) = \sum_{d|m} F(m/d) \sum_{d'|d} g(d/d')S(d')$$
$$= \sum_{d|m} (m/d)Q(d)\cdot dQ''(d) \equiv 0 \pmod{m}$$

(Q, Q'' integer-valued).

In particular, for $R \equiv \pm 1$ Lemma 2 implies

LEMMA 3. *If $f(m)$, $g(m)$, $S(m)$ are three functions satisfying the conditions*

$$g(1) = \pm 1, \tag{4'}$$

$$\sum_{d|m} g(d) \equiv 0 \pmod{m}, \tag{5'}$$

$$\sum_{d|m} f(d) \equiv 0 \pmod{m} \tag{6'}$$

and (8), *then also* (7') *hold.*

Proof of Theorem 1 and Corollary. The theorem and corollary follow by Jänichen's criterion from Lemma 3 by first, putting μ for g and, second, putting f for g and μ for f.

CONVERSE OF THEOREM 1. *Suppose that f is a number-theoretic function such that every set S has the property P if and only if* (1) *hold. Then f satisfies* (2) *and* (3').

Proof. (3') is obvious taking $p(x) = x-1$, whence $S(m) \equiv 1$. To prove (2), suppose $f(1) \neq \pm 1$. Then there exists a prime π dividing $f(1)$. Now we can choose a set S which satisfies (1) but does not have the property P. For example $n = \pi$, $S(m) = 0$ for $m = 1, 2, \cdots, \pi-1$, $S(\pi) = 1$, which implies that the congruences (1) hold for each $m \leqq \pi$. But, by Jänichen's criterion, this set S does not have the property P, since we have $\mu(1)S(\pi) + \mu(\pi)S(1) = 1 \not\equiv 0 \pmod{\pi}$.

3. The apparition of prime factors. The aim of the second part of this note is to prove some theorems on the apparition of prime factors in sequences (s_q). From the criterion of Jänichen one can, with I. Schur [3], deduce the following congruences:

$$s_{kp^{\alpha+1}} \equiv s_{kp^\alpha} \pmod{p^{\alpha+1}} \tag{9}$$

for every prime p and non-negative integral α. The congruences (9) can also be written in the following equivalent form:

$$s_{kp^{\alpha+\beta}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}} \tag{10}$$

for every positive integral β. Indeed, (10) becomes (9) for $\beta=1$. Let (10) be true for β. Then by (9)

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha+\beta}} \pmod{p^{\alpha+\beta+1}},$$

and a fortiori

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha+\beta}} \pmod{p^{\alpha+1}},$$

Combining the last congruences with (10), supposed true for β, we have

$$s_{kp^{\alpha+\beta+1}} \equiv s_{kp^{\alpha}} \pmod{p^{\alpha+1}},$$

that is, (10) is true also for $\beta+1$, which establishes (10).

From (10) we immediately deduce the following theorems:

THEOREM 2. *If $s_{kp^{\alpha}} \equiv 0 \pmod{p^{\gamma}}$, where p is a prime, γ is a positive integer, α is a non-negative integer and $\gamma < \alpha+1$, then $s_{kp^{\alpha+\beta}} \equiv 0 \pmod{p^{\gamma}}$ for every positive integer β.*

In particular, for $\alpha=0$, $\gamma=1$ we have

THEOREM 2.1. *If $s_k \equiv 0 \pmod{p}$, where p is a prime, then $s_{kp^{\beta}} \equiv 0 \pmod{p}$ for every positive integer β.*

THEOREM 2.2. *If $s_k = 0$ then $s_{kp^{\beta}} \equiv 0 \pmod{p}$ for every prime p and every positive integer β* [4].

CONVERSE OF THEOREM 2.2. *If $s_k \equiv 0 \pmod{p}$ for an infinitude of primes p then $s_k = 0$.*

Proof. By (10) we have $s_{kp} \equiv s_k \pmod{p}$. Whence, by the hypothesis of the converse, $s_k \equiv 0 \pmod{p}$ for an infinitude of primes p. Thus $s_k = 0$.

4. Remarks. A sequence (s_q), no term of which vanishes, does not necessarily contain all primes as factors. For example, the sequence $(V_q = \alpha^q + \beta^q)$, where α, β are roots of the equation $x^2 - x - 1 = 0$ ($V_1 = 1$, $V_2 = 3$, $V_q = V_{q-1} + V_{q-2}$) does not contain all primes as factors [5]. The question as to which primes appear and which do not appear as factors in a sequence (s_q) with no vanishing term seems to be open even in the case of (V_q).

The Theorems 2 and 2.1 can be generalized immediately by putting r for 0 in the congruences.

References

1. W. Jänichen, Über die Verallgemeinerung einer Gauss'schen Formel aus der Theorie der höheren Kongruenzen, Sitzungsberichte der Berliner Mathematischen Gesellschaft 20 (1921), 23–29.

2. The expressions $\sum_{d|m}\mu(d)s_{m/d}$, $\sum_{d|m}\phi(d)s_{m/d}$ have an interpretation in the theory of ordered partitions. Compare Th. Motzkin, Ordered and Cyclic Partitions, Riveon Lematematika 1 (1946–1947), 61–67.

Originally the special case of Theorem 1 where f is Euler's function ϕ was proved. I am indebted to the referee for pointing out the possibility of further generalization.

3. I. Schur, Arithmetische Eigenschaften der Potenzsummen einer algebraischen Gleichung, Compositio Mathematica 4 (1937), 432–44.

4. The special case $k=\beta=1$ was found by L. E. Dickson, this MONTHLY 15 (1908), 209. Particular cases of his result appear already in E. Lucas, A. F., Congrès de Clermont-Ferrand, (1876), 2 (according to Escott, L'interméd. des Math. 8 (1901), 63); R. Perrin, L'interméd. des Math. 6 (1899), 76–77; E. Malo, ibid. 7 (1900), 280–282, 321–314; E. B. Escott, ibid. 8 (1901), 63–64.

5. E. Lucas, Amer. Journ. of Math. 1 (1878), 298.

HUGH JONES AND OCTAVE COMPUTATION

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Many mathematicians have at diverse times indulged themselves in the contemplation of a change in the base of the number system. So far as the author has been able to discover the first person in this country to advocate it seriously, at length, and with missionary zeal, was the Rev. Hugh Jones. This gentleman (1692–1760), graduate of Jesus College, Oxford, clergyman, mathematician, and historian, left England in 1716 shortly after securing the Master of Arts degree. In 1717 he was elected to the faculty of the College of William and Mary as successor to the chair of mathematics and natural philosophy first held by the Rev. Tanaquil Lefevre. He was in consequence the second professor of mathematics in the territory now known as the United States, antedating by ten years the appointment of Isaac Greenwood to the Hollis professorship at Harvard in 1727.

Jones was learned, fearless, aristocratic, a loyal Hanoverian, and a zealous churchman. In addition he was intellectually vigorous as is evidenced by the fact that aside from his duties as professor, chaplain of the House of Burgesses, and lecturer at Bruton Church in Williamsburg, he wrote an *Accidence to Christianity*, an *Accidence to the Mathematicks*, an *Accidence to the English Tongue* (the first English Grammar written in America), and a most interesting book upon the *Present State of Virginia*. This latter work was not only sprightly and interesting but withal an invaluable source of material for subsequent historians. In it he advanced many sound suggestions for the improvement of the colony, among them a proposal for a separate chair of history for the College of William and Mary.

One of his practically unknown pieces of research is a long manuscript, listed as British Museum Additional MS 21893, entitled *The Reasons, Rules and Uses of Octave Computation or Natural Arithmetic.* A very abbreviated resumé of the

material appeared in the Gentleman's Magazine of March, 1751, but this was known to only a handful of historians interested in the detailed history of Virginia. The original manuscript, Most humbly inscribed to the Right Honorable Earl of Macclesfield, consists of forty pages, 7×9 inches, excellently phrased and boldly executed in a chirography not unlike the familiar signature of John Hancock appended to the Declaration of Independence. There is evidence that the final preparing of the manuscript was the work of some amanuensis since it is not the handwriting of Hugh Jones and is considerably embellished throughout.

Professor Jones observes at the outset that since there is nothing of more general utility than a due adjustment of numbers, coins, measures, and weights it appears desirable to scrutinize their properties and consider methods for their uniform representation, particularly since nations differ so much in these matters that even common arithmetic has become "mysterious to Women and Youths and often troublesome to the best Artists."

The argument moves on to assert that much of the difficulty lies in the choice of ten as a base of the number system, especially since neither ten, nor its powers or products, admit of bi-partition down to unity. Furthermore neither ten, nor its half or its double, have square or cube roots that are integers, "all of which properties the common Radix of Number ought to have for convenience in applications to surfaces and solids."

The elimination of many of the difficulties is alleged to lie in the choice of eight for the "Radix of Computation." According to the author,

> For 8 its Powers and many of its Products are divisible down to Unity by halving, quartering etc. And the Proportion of its Parts to the whole are self-evident, without Perplexity to the Thought; for 1 is the half quarter; 2 the Quarter; 3 the Quarter and half Quarter; 4 the Half; 5 the Half and half Quarter; 6 the three Quarters; 7 the three Quarters and half Quarter of 8: Into which Portions a geometrical Line Surface or Solid is most easily and readily and precisely divisible; and therefore to this Natural Method Numbers ought to conform. Again 8 is a cubic Number whose Root is 2; 4 its Half is a Square whose Root is likewise 2; 16 its Double is a square Number whose whole Root is 4; and 64, besides being the Square of 8, is likewise a cubic Number whose Root is 4.
>
> So that Arithmetic by *Octaves* seems most agreeable to the Nature of Things, and therefore may be called Natural Arithmetic in Opposition to that now in Use, by Decades; which may be esteemed Artificial Arithmetic.

The above thesis is then buttressed by several pages of applications to arithmetic, geometry, and natural philosophy.

The second main division of the manuscript is entitled "Notation and Numeration." With considerable ingenuity the term "ones" is retained for the

units while the successive powers of the base are designated as ers, ests, thousets, millets, billets, etc. In other words, 8×8 is the est of 8 while $8\times8\times8$ is the thouset, and so on to any desired magnitude. In this connection it is the opinion of the writer that the names beginning with thouset are inconsistent since they derive from powers of ten. Mr. Jones, however, in his enthusiasm goes on to suggest that additional clarification would result by applying his suffixes to classification names, as for example, cash, casher, cashest in counting money, or ounce, ouncer, ouncest in weighing. As an example 352 yardest would signify $3\times8^2+5\times8+2$ yards.

Following this material there appears the technique for changing from base ten to base eight, familiar to any experienced mathematician. For the layman unacquainted with the process it must first be pointed out that in an octave number system eight distinct symbols, and only eight, namely 0, 1, 2, 3, 4, 5, 6, 7, or their equivalents, are required to write any integer whatsoever. With this brief word of explanation we may proceed directly to the examples in the manuscript.

$1 \times 8 =$ er $\qquad 2 \times 4 = 10 =$ er, where 10 means not ten but one times the base plus no units,

$2 \times 5 = 12 =$ er2, $\qquad 2 \times 6 = 14 =$ er4,

$2 \times 7 = 16 =$ er6, $\qquad 2 \times 8 = 20 =$ 2er,

$8 \times 8 = 100 =$ est, $\qquad 6 \times 6 = 44 =$ 4er4,

$7 \times 7 = 61 =$ 6er1

To identify the process outlined above the author coins the verb "octavate." For instance, to octavate the number one hundred the reasoning is as follows. The square of eight goes into one hundred once with remainder thirty-six. This remainder contains eight to the first power four times with remainder four. Hence to express one hundred to the base 8 we write the digits 144 which in somewhat longer form may be put as $100=1\times8^2+4\times8+4$. The inverse operation, which is termed "decimation," together with an adequate treatment for the octavation of decimal fractions will be mentioned here without consideration of the details.

The thread of the argument then turns to the advantages which an octave system could effect in the matter of coinage and in this connection the following query is set forth:

> Might not this Division be easily effected by altering the Proportion of Silver to Copper by 1/16 and calling the present Half Penny a Penny as it is already called in these Plantations? Then our present Half Crown would contain 64 (instead of 60) of these Pennies; and the Pound or Integer would be equal to 512 of them, & instead of 960, there would be esteemed 1024 of our Farthings in a Pound Sterling. May not the vast Quantity of Copper lately found in our Plantations on the Continent

> of *North America* well allow (if not require) this Alteration in the Proportion of the Value of Silver to that Metal?

In discussing the matter of physical units Mr. Jones advances several interesting notions. By hanging a weighted cord between two cycloids the concept of the isochronous pendulum is introduced, but in addition he defines the unit of linear measure as the length of such a pendulum which at sea level at Dover or Calais beats seconds. Convenient octave submultiples may then be taken as experience proves desirable. From this he then makes a suggestion which should accord him a high place among the scientific thinkers of his day, namely, to derive the unit of weight from the amount of rain water contained in a cubical box with edge equal to one eighth of the pendulum length above mentioned. This idea was to be actually established in its decimal counterpart forty-five years later when the French National Assembly took up the matter of the metric system in 1790.

In spite of the zeal and meticulousness of the author he is not unmindful of the inertia of human nature with respect to new ideas since he pauses to observe,

> But forasmuch as there seems no Probability that this will be soon, if ever, universally complied with; therefore it may not be impertinent to subjoyn some Methods that appear easiest for the Reduction of our present Denominations of Measures and Weights to an octave Uniformity in their different Kinds.

In this connection the term "Longimetry" is introduced to designate the process of measuring "mere length" and the observation is made that such current units as fathom, pace, ell, yard, foot, league, mile, furlong, and perch are in several instances already commonly divided into eighths. In particular is this true of the foot where the octave system would produce a unit, called a "Prime," equivalent to the present inch and a half. One eighth of this is designated as a "Second" and an additional subdivision by eight is suggested but is accorded no name.

The diurnal rotation of the earth is termed the "Nuethemeron," and is divided into eight "Primes," and these into eight "Seconds" (each equivalent to twenty-two and a half of the present minutes). This idea is not pursued to its logical conclusion but it would lead to a further division by sixty-four in order to secure a unit of the order of magnitude of the present second in which case it would then seem necessary to redetermine the length of the pendulum and other derived units previously mentioned.

The final section of the manuscript deals with additional utilitarian applications to land surveying, grain storage, and ship displacement after which the author polishes off the whole paper with the following paragraph which may or may not contain elements of unconscious humor:

> Much more might be added on these Subjects; but what is already advanced being more than was at first intended, and perhaps too prolix al-

ready, seeming sufficient to explain the Scheme proposed in the Gentleman's Magazine for July 1745 (p. 377) and publickly demanded in the first page of the Gent. Magaz. for March 1751 it is thought needless to expatiate at present; since, if these Lucubrations meet with favorable Reception, and require further Illustration or Proof, an Enlargement may be communicated by the Proposer.

MATHEMATICAL NOTES

EDITED BY E. E. BECKENBACH, University of California and Institute for Numerical Analysis of the National Bureau of Standards

Material for this department should be sent to E. F. Beckenbach, University of California, Los Angeles 24, California.

A TOPOLOGICAL PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA*

B. H. ARNOLD, Oregon State College

Many theorems which were formerly thought to be purely analytic in character have in recent years been associated with topological concepts. For instance, it has been known for some time that the fundamental theorem of algebra could be derived from Brouwer's fixed point theorem. Such a derivation does not appear in the literature and it is the purpose of this note to supply a simple proof. Proofs have appeared [2, 3] which are based on the Brouwer index, on which the fixed point theorem rests.

For our purpose, Brouwer's fixed point theorem may be stated: Any continuous transformation of the circle $|z| \leqq R$ into itself has a fixed point. Here R is an arbitrary positive real number (to be specified later) and z is a complex variable. For an elementary proof of this theorem, see [1].

To derive the fundamental theorem of algebra from this result, let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be any complex polynomial with leading coefficient unity. We must prove that $f(z)$ has at least one zero. To this end, set $z = re^{i\theta}$, $0 \leqq \theta < 2\pi$, $R = 2 + |a_1| + \cdots + |a_n|$ and define a (non-analytic) function $g(z)$ by

$$g(z) = \begin{cases} z - f(z)/(Re^{i(n-1)\theta} r) & \text{for } |z| \leqq 1 \\ z - f(z)/(Rz^{n-1}) & \text{for } |z| \geqq 1. \end{cases}$$

The function $g(z)$ is single valued and continuous for all values of z. First, each of the two expressions given for $g(z)$ is continuous throughout the range

* Published with the approval of the Oregon State College Monographs Committee, Research Paper No. 138, Department of Mathematics, School of Science.

specified since neither denominator becomes zero and, for $z=0$, we have $r=0$ so that $i(n-1)\theta r=0$ no matter what value θ has. Secondly, for $|z|=1$, the two expressions are identical.

For $|z|\leqq R$ we have $|g(z)|\leqq R$. First, for $|z|\leqq 1$,

$$|g(z)|\leqq |z|+\left|\frac{f(z)}{R}\right|\leqq 1+\frac{1+|a_1|+\cdots+|a_n|}{R}\leqq 1+1\leqq R.$$

Secondly, for $1\leqq|z|\leqq R$,

$$|g(z)|=\left|z-\frac{z}{R}-\frac{a_1+\cdots+a_nz^{1-n}}{R}\right|\leqq\left|(R-1)\frac{z}{R}\right|$$
$$+\frac{|a_1|+\cdots+|a_n|}{R}\leqq R-1+\frac{R-2}{R}\leqq R.$$

Now consider the correspondence $z\to g(z)$. The last two paragraphs show that this is a continuous transformation which maps the circle $|z|\leqq R$ into itself. Thus, by Brouwer's fixed point theorem, there exists at least one value z_0 such that $g(z_0)=z_0$. But from the definition of $g(z)$, this means that $f(z_0)=0$ and $f(z)$ has at least one zero.

Incidentally this proof shows the existence of a zero of $f(z)$ whose absolute value does not exceed R, but this bound is of little interest since much stronger results are known. Goursat [4] gives a result which implies that *all* the zeros of $f(z)$ are less than or equal to max $(1, R-2)$ in absolute value.

References

1. R. Courant and H. Robbins, What is Mathematics?, Oxford University Press (1941) pp. 251–254.

2. Ibid. pp. 269–271.

3. S. Eilenberg and I. M. Niven, The "fundamental theorem of algebra" for quaternions, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 246–248.

4. E. Goursat, Mathematical Analysis, vol. II—Part I, Functions of a Complex Variable, Ginn & Co. (1916), p. 104.

ON THE ALGEBRAIC STRUCTURE OF KNOTS*

FRANK HARARY, University of Michigan

1. Motivation of the postulates. By definition, a knot is a closed polygon in 3 dimensional Euclidean space. If k_1 and k_2 are knots, then their product knot $k_1\times k_2$ is the knot obtained by cutting k_1 at any point P_1 (giving loose ends e_1 and f_1), cutting k_2 at any point P_2 (giving loose ends e_2 and f_2), and joining e_1 to e_2, and f_1 to f_2. Pictorially, the product $k_1\times k_2$ is the knot of Figure 1 when k_1, k_2 are the knots of Figure 2 on the left and right, respectively. It has been shown that

* This note was motivated by a Colloquium address at the University of Michigan by Professor H. Seifert, now at the University of Heidelberg. Section 1 is essentially a summary of part of his lecture.

the product of knots is commutative and associative. Obviously the circle is the identity knot. Let c denote the circle. By definition, k is a prime knot if $k=k_1\times k_2$ implies $k_1=c$ or $k_2=c$. Let $g(k)$ denote the genus of the knot k. Then it is known that

(1) $$g(k_1 \times k_2) = g(k_1) + g(k_2),$$

(2) $g(k)$ is a non-negative integer.

The rather spectacular result has recently been shown (by Schubert) that every knot can be uniquely factored into a product of prime knots.

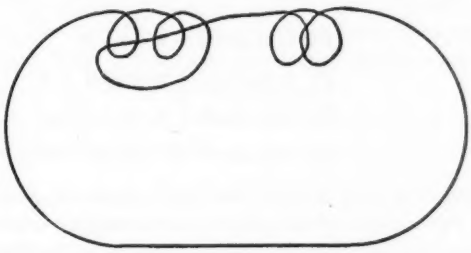

Fig. 1

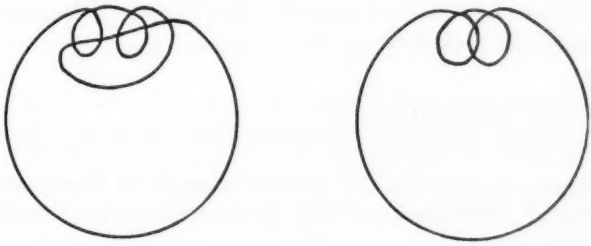

Fig. 2

2. Postulates for knots. From the above considerations, it follows that the set K of all knots satisfies the following abstract postulates with respect to the binary operation of knot multiplication.

Let the undefined notions be a non-empty class K of elements and a binary operation denoted by $\times$. The postulates are:

*P*1: If $k, k'\in K$, then $k\times k'\in K$.
*P*2: If $k, k', k\times k', k'\times k\in K$, then $k\times k'=k'\times k$.
*P*3: If $k, k', k'', k\times k', k'\times k'', k\times(k'\times k''), (k\times k')\times k''\in K$, then $k\times(k'\times k'')=(k\times k')\times k''$.
*P*4: There exists an element $c\in K$ such that if k and $c\times k\in K$, then $c\times k=k$.

Remark: If we add a postulate $P5$: If $k \in K$, then there is an element $k^{-1} \in K$ such that $k \times k^{-1} = c$; then $P1$ to $P5$ are a set of postulates for an abelian group.

3. Independence of the postulates. The following finite systems show the independence of postulates $P1$, $P2$, $P3$, $P4$:

$P1$

$\times$	0	1
0	0	1
1	1	2

$P2$

$\times$	0	1
0	0	1
1	0	1

$P3$

$\times$	0	1	2
0	0	1	2
1	1	1	0
2	2	0	2

$P4$

$\times$	0	1
0	0	0
1	0	0

$P1$ fails since $1 \times 1 \notin K$

$P2$ fails since $1 \times 0 \neq 0 \times 1$

$P3$ fails since $1 \times (1 \times 2) \neq (1 \times 1) \times 2$

$P4$ fails since there is no identity element.

4. Conclusions. It is well known that any algebraic system satisfying the postulates $P1$ to $P4$ (which can be called a commutative semi-group with identity element, if we choose that definition of the many extant for semi-groups which characterizes a semi-group as a closed associative system) can be embedded in an abelian group. Thus if $(K, \times)$ is the algebra of knots (i.e. $K =$ the set of all knots, and $k_1 \times k_2$ is knot product), then $(K, \times)$ can be embedded in an abelian group $(\overline{K}, \otimes)$ such that:

1. $K \subset \overline{K}$.
2. If $k, k' \in K$, then $k \otimes k' = k \times k'$.
3. For each $k \in K$, there exists an element $k^{-1} \in \overline{K} - K$ such that $k \otimes k^{-1} = c$.

It is interesting to observe that $\overline{K} - K$ would consist of "imaginary knots" of "negative genus." The element k^{-1} can be pictured as that "knot" which *unravels* a given real knot k.

CLASSROOM NOTES

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MEAN AND ORDINARY CONVERGENCE OF A SEQUENCE OF FUNCTIONS

D. L. THOMSEN, Haverford College

In presenting the idea of mean convergence to undergraduates who are not acquainted with the Lebesgue integral the following procedure has been found to be helpful. We point out that mean convergence may apply to cases where ordinary convergence fails, and we also prove that under appropriate conditions ordinary convergence implies mean convergence.

A sequence of functions $f_n(x)$, defined in the finite closed interval (a, b), converges to $f(x)$ in the ordinary sense if

$$\lim_{n\to\infty} f_n(x) = f(x). \tag{1}$$

A sequence of functions $f_n(x)$ converges in the mean with index $p>0$ to $f(x)$ if

$$\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p \, dx = 0. \tag{2}$$

Throughout the discussion we assume the integration to be taken in the Riemann sense.

Convergence in the mean does not imply ordinary convergence. A familiar example* is the following. Consider the closed intervals $(0, 1/2)$, $(1/2, 1)$, $(0, 1/3)$, $(1/3, 2/3)$, $(2/3, 1)$, $(0, 1/4)$, $\cdots$. Let $f_n(x)=1$ in the nth interval and zero elsewhere. Here $\lim_{n\to\infty} f_n(x)$ does not exist. But the limit in the mean does exist since we have $\lim_{n\to\infty} \int_0^1 |f_n(x)-f(x)|^p \, dx = \lim_{n\to\infty} L_n = 0$ where L_n is the length of the nth interval and $f(x)=0$.

However ordinary convergence, uniform or non-uniform, does imply mean convergence under the conditions as stated in the following theorem.

THEOREM. *If in the finite closed interval (a, b) $f_n(x)$ is bounded in both n and x, if $f_n(x)$ and $f(x)$ are Riemann integrable, if $|f_{n+1}-f(x)| \leqq |f_n(x)-f(x)|$, and if $\lim_{n\to\infty} f_n(x) = f(x)$, then $\lim_{n\to\infty} \int_a^b |f_n(x)-f(x)|^p \, dx = 0$ $(p>0)$.*

Proof. Let $F_n(x)=f_n(x)-f(x)$, and let $|f_n(x)| \leqq M/2$ so that $|f(x)| \leqq M/2$. Then we have $|F_n(x)| \leqq M$ for all x and n, and the $\lim_{n\to\infty} F_n(x)=0$. Let $I_n = \int_a^b |F_n(x)|^p \, dx$. We must show $\lim_{n\to\infty} I_n = 0$. For a fixed $\epsilon>0$ and an n we have either $|F_n(x)| < \epsilon$ or $\epsilon \leqq |F_n(x)| \leqq M$. These two inequalities divide (a, b) into two sets of intervals. We may ignore the isolated singularities of $F_n(x)$ in making this subdivision since they do not affect the value of the integral. Thus we

* Titchmarsh: The Theory of Functions, Art. 12.53

have $I_n < (b-a)\epsilon^p + M^p B_n$ where B_n is the sum of the lengths of all intervals in which $|F_n(x)| \geqq \epsilon$. But the $\lim_{n\to\infty} B_n = 0$ as may be seen by considering the sequence, $B_1, B_2, B_3, \cdots, B_n, \cdots$. We necessarily have $B_1 \geqq B_2 \geqq B_3 \geqq \cdots \geqq B_n \geqq \cdots$ in view of the fact that $|F_{n+1}(x)| \leqq |F_n(x)|$. Such a positive sequence must necessarily approach a unique limit since it is monotonic, is bounded above by $(b-a)$, and is bounded below by zero. Let this limit be equal to $B \neq 0$. Then we have $|F_n(x)| \geqq \epsilon$ for all n in intervals whose sum is B. This means there are points where $\lim_{n\to\infty} |F_n(x)| \neq 0$, which is contrary to hypothesis. Thus $B = 0$. Hence I_n may be made as small as desired, and the theorem is proved. The theorem may be extended to sequences which are not monotonic provided the intervals over which the sequences are not monotonic can be made as small as desired. This is actually the case in the first example below.

Example 1. The function $f_n(x) = n^c x^{1/2} \exp(-n^2x^2/4)$ in $(0, 1)$ with $f(x) = 0$ illustrates the behavior of the two types of convergence. The $\lim_{n\to\infty} f_n(x)$ is zero everywhere for all c. The maximum value of $f_n(x)$ occurs at $x = 1/n$ where we have $f_n(1/n) = n^{c-1/2} \exp(-1/4)$. The convergence of the sequence $f_n(x)$ is uniform for $c < 1/2$. Our theorem now tells us we have mean convergence when $c \leqq 1/2$ providing we note the comment at the end of the proof above; for we have $f_n(x) \geqq f_{n+1}(x) \geqq f_{n+2}(x) \geqq \cdots$ except in an interval of length less than $1/n$. Now by direct integration we may verify mean square convergence ($p = 2$), for we have $\int_0^1 |f_n(x) - f(x)|^2 \, dx = n^{2c-2}(1 - \exp(-n^2/2))$. The table below shows the various possibilities for $\max f_n(x)$ and mean square convergence as c increases.

	$\lim_{n\to\infty} f_n(1/n)$	$\lim_{n\to\infty} \int_0^1 \lvert f_n(x) - f(x) \rvert^2 dx$
$c < 1/2$	0	0
$c = 1/2$	$\exp(-1/4)$	0
$1/2 < c < 1$	∞	0
$c = 1$	∞	1
$c > 1$	∞	∞

Example 2. Let $f_n(x) = \exp(nx^2)/(1+\exp(nx^2))$ in $(-1, 1)$ where $f(0) = 1/2$ and $f(x) = 1$ when $x \neq 0$. Both $f_n(x)$ and $f(x)$ satisfy the hypothesis of the theorem above, and hence we have mean convergence. Here the "lim" cannot pass under the integral sign because the integrand does not converge uniformly; also, it is impossible to perform the integration by elementary methods.

We may still have mean convergence when $f_n(x)$ is unbounded or when (a, b) is an infinite interval. If we consider mean square convergence, always we must have the area bounded by $[F_n(x)]^2$ and the x-axis arbitrarily small. For example,

the function $f_n(x)$ may be unbounded in n as in Ex. 1 above when $1>c>1/2$, unbounded in x (as in the sequence $(nx)^{-1/3}$ in $(-1, 1)$ where $f(0)=\infty$ and $f(x)=0$ for $x\neq 0$), or unbounded in both x and n (as in a suitable combination of the two preceding cases). On the other hand consider $f_n(x)=(nx)^{-3/5}$. This sequence has the same limit function as $(nx)^{-1/3}$ above; the integral $\int_{-1}^{1} f_n(x)\,dx$ converges absolutely as an improper integral; but $f_n(x)$ does not converge to $f(x)$ in the mean square sense in $(-1, 1)$.

PROOFS OF THE ADDITION FORMULAE FOR SINES AND COSINES

A condensation by the editor of independent papers by

L. J. BURTON, Bryn Mawr College, and E. A. HEDBERG, University of South Carolina

The standard proof of the addition formulae for sines and cosines is certainly one of the least satisfactory sections of the usual trigonometry text. Its chief failings are its complexity, the artificiality of the construction required, and the limitation of the magnitudes of the angles involved. Two alternative proofs of these formulae are published below. The first, submitted by L. J. Burton, assumes a very elementary knowledge of analytic geometry. This proof has been taught in several universities for some time and probably has a long history; but

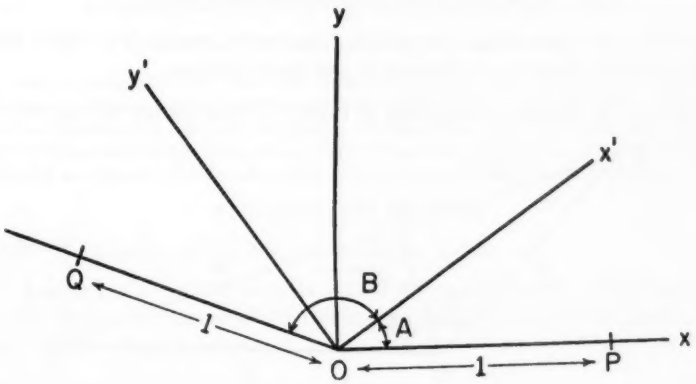

FIG. 1

since it does not appear in the popular textbooks and is unknown to most teachers, it seems desirable to make it more generally available. It has the advantage of placing no restriction on the size of the angles in addition to greater simplicity. The second, submitted by E. A. Hedberg, is based upon the laws of sines and cosines and assumes no analytic geometry. Since it makes use of triangles, it is valid only in case all of the angles involved are less than 180 degrees.

Proof by L. J. Burton: To prove that:

$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

Referring to Figure 1, place angle A so that its vertex is at the origin and its initial side is along the x-axis; place angle B so that its vertex is at the origin and its initial side coincides with the terminal side of angle A. Choose points P and Q at a unit distance from the origin on the initial side of A and the terminal side of B respectively. Consider also a new coordinate system with the same origin and with its x'-axis along the terminal side of A (which is also the initial side of B). The coordinates of P and Q in the two coordinate systems are seen to be:

	xy system	$x'y'$ system
P	$(1, 0)$	$[\cos (-A), \sin (-A)] = (\cos A, -\sin A)$
Q	$[\cos (A + B), \sin (A + B)]$	$(\cos B, \sin B)$

The distance PQ in the two coordinate systems is given by:

$$xy \text{ system:} \quad PQ^2 = [\cos (A + B) - 1]^2 + \sin^2 (A + B)$$
$$= 2 - 2 \cos (A + B)$$
$$x'y' \text{ system:} \quad PQ^2 = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$$
$$= 2 - 2 \cos A \cos B + 2 \sin A \sin B.$$

By equating the two results we obtain the desired result. The other addition formulae are obtained from this one in the usual fashion.

Proof by E. A. Hedberg: Referring to Figure 2, from the law of cosines:

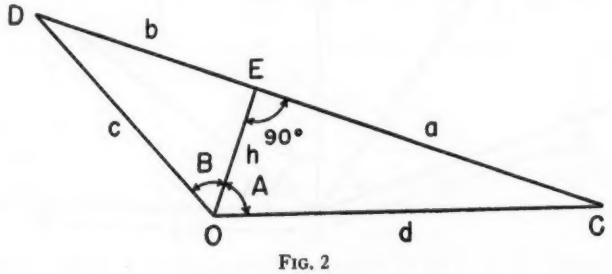

FIG. 2

$$\cos (A + B) = \frac{c^2 + d^2 - (a + b)^2}{2dc}$$
$$= \frac{(c^2 - b^2) + (d^2 - a^2) - 2ab}{2dc}$$
$$= \frac{2h^2 - 2ab}{2dc} = \frac{h^2 - ab}{dc}$$
$$= (h/d)(h/c) - (a/d)(b/c).$$

Thus, $\cos (A + B) = \cos A \cos B - \sin A \sin B.$

The addition formula for the $\sin (A + B)$ may also be obtained from Figure 2, for by the law of sines:

$$\begin{aligned}\sin (A + B) &= \frac{(a + b) \sin D}{d} \\ &= (a/d) \sin D + b(\sin D/d) \\ &= (a/d) \sin D + b(\sin C/c) \\ &= (a/d) \sin D + (b/c) \sin C \\ &= \sin A \cos B + \cos A \sin B.\end{aligned}$$

The subtraction formulae can be obtained from these or directly from a modification of Figure 2 and a similar technique.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 876. *Proposed by H. L. Lee, University of Tennessee*

The product of four consecutive terms of an arithmetic progression of integers plus the fourth power of the common difference is a perfect square but in no case a perfect fourth power.

E 877. *Proposed by L. J. Burton, Bryn Mawr College*

Each of three arithmetic progressions continues indefinitely in both directions, and each has a difference which is an integer. Prove that if there is a term common to each pair of progressions then there is a term common to all three progressions.

E 878. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Let S be the incenter of the right triangle ABC, and X the point of contact of the hypotenuse BC with the incircle. With center X and radius XS describe the circle cutting BS, CS at M and N respectively. Let AD be the altitude on

BC. Show that M, N are the incenters of the right triangles ABD and ACD respectively.

E 879. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Let S_1, S_2, S_3 be the midpoints of three concurrent cevians of triangle ABC. Let S_2S_3, S_3S_1, S_1S_2, meet the sides BC, CA, AB in A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3 respectively. Show that (1) A_2, A_3; B_3, B_1; C_1, C_2 are isotomic points on the segments BC, CA, AB, (2) A_1, B_2, C_3 are collinear, (3) A_2, A_3, B_3, B_1, C_1, C_2 lie on a conic.

E 880. *Proposed by Peter Ungar, University College, London, England*

Let n points be given in the plane, not all on a straight line. The shortest closed route connecting them is a simple polygon.

SOLUTIONS

The Sum of a Series

E 844 [1949, 31]. *Proposed by Orrin Frink, Pennsylvania State College*

Sum the series

$$1 + 1/5! + 1/10! + 1/15! + \cdots + 1/(5n - 5)! + \cdots .$$

I. *Solution by W. Fulks, University of Minnesota.* Consider the function

$$f(x) = \sum_{n=0}^{\infty} x^{kn}/(kn)!,$$

where k is a positive integer. We note that $f(x)$ satisfies the system $f^{(k)}(x) = f(x)$, $f(0) = 1$, $f'(0) = f''(0) = \cdots = f^{(k-1)}(0) = 0$. The unique solution of this system is

$$f(x) = (1/k)\sum_{j=1}^{k} \exp\,(w_k^j x),$$

where w_k is a primitive kth root of unity. Thus the required sum is, taking $x = 1$, $k = 5$,

$$S = (1/5)\sum_{j=1}^{5} \exp\, w_5^j.$$

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* Since the sum of the mth powers of the k kth roots of unity is zero unless m is a multiple of k, in which case the sum is unity, we have, from the Maclaurin expansion of e^x,

$$\sum_{j=1}^{k} \exp\,(w_k^j x) = k\sum_{n=0}^{\infty} x^{kn}/(kn)!,$$

where w_k is a primitive kth root of unity. The required sum, obtained by taking $x=1$, $k=5$, is then

$$S = (1/5)\sum_{j=1}^{5} \exp w_5^j.$$

Also solved by W. G. Brady, D. H. Browne, Paul Carnahan, P. L. Chessin, Ragnar Dybvik, J. L. Ericksen, Harley Flanders, H. E. Gould, Roger Lessard, D. C. B. Marsh, Leo Moser, C. S. Ogilvy, S. T. Parker, C. F. Pinzka, Alex Rosenberg, Azriel Rosenfeld, C. M. Sandwick, F. C. Smith, W. R. Talbot, an unsigned solver, and the proposer.

Several solvers easily reduced the above sum to

$$S = (1/5)[e + 2e^{\cos 72^\circ}\cos(\sin 72^\circ) + 2e^{-\cos 36^\circ}\cos(\sin 36^\circ)].$$

If we are merely interested in obtaining a numerical result, however, there is little point of transforming the original series, which converges very rapidly. Thus four terms of the series gives $S=1.00833360890$.

Smith picked up, as a by-product, the pretty summations

$$\sum_{n=0}^{\infty} (x^n \cos n\theta)/n! = e^{x\cos\theta}\cos(\sin\theta),$$

$$\sum_{n=0}^{\infty} (x^n \sin n\theta)/n! = e^{x\cos\theta}\sin(\sin\theta).$$

A Difficult Enumeration Problem

E 845 [1949, 31]. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

It is required to write the fifteen combinations of a, b, c, d in a sequence such that any two adjacent terms of the sequence shall differ by a single letter. How many such sequences are there? How can they be written down?

Partial solution by the Proposer. It is obvious that if a certain sequence of the combinations satisfies the requirement, then the corresponding sequence resulting from any permutation of a, b, c, d will also satisfy the requirement. Such a pair of sequences will be called *dependent*, and in what follows only *independent* sequences will be considered.

We shall now show how a number of the required independent sequences can be written down.

If a sequence C of the combinations satisfies the requirement, and if its first term is a single letter, then corresponding to C we shall associate the following sequence S of fifteen terms. The first term of S will be the same as the first term of C, and every subsequent term of S will be that letter in which the corresponding term of C differs from its preceding term. Thus the terms of S are single letters, and it is obvious how the sequence C can be written down when its associated S is given.

In view of the above it will suffice to show how to obtain a number of valid S sequences. For this purpose it is observed that if $S(n)$ is a sequence for n letters $a_1, a_2, \cdots, a_n$ whose corresponding sequence $C(n)$ satisfies the requirement of the problem, then the S sequence for $n+1$ letters $a_1, a_2, \cdots, a_n, a_{n+1}$, defined by

$$S(n+1) = S(n),\ a_{n+1},\ S(n), \tag{1}$$

will be a valid sequence for the $n+1$ letters. More generally, if $S_1(n)$ and $S_2(n)$ are each valid sequences for $a_1, a_2, \cdots, a_n$, then the sequence defined by

$$S(n+1) = S_1(n),\ a_{n+1},\ S_2(n) \tag{2}$$

is a valid sequence for $a_1, a_2, \cdots, a_n, a_{n+1}$.

Now, starting with the two sequences $S_1(2) = a_1, a_2, a_1$ and $S_2(2) = a_2, a_1, a_2$, even though they are not independent, there are obtained by means of relations (1) and (2) the two independent sequences

$$S_1(3) = a_1, a_2, a_1, a_3, a_1, a_2, a_1, \tag{3}$$

and

$$S_2(3) = a_1, a_2, a_1, a_3, a_2, a_1, a_2. \tag{4}$$

In addition it can be verified that

$$S_3(3) = a_1, a_2, a_3, a_2, a_1, a_2, a_3 \tag{5}$$

is also a valid sequence.

Next, in each of $S_1(3)$, $S_2(3)$, $S_3(3)$, write all the six sequences corresponding to the six permutations of a_1, a_2, a_3. Denoting the resulting sequences by A_i, B_i, C_i, $i=1, \cdots, 6$, there are obtained the eighteen (not independent) sequences $S(3)$:

$$\begin{array}{lll}
A_1 = a_1a_2a_1a_3a_1a_2a_1 & B_1 = a_1a_2a_1a_3a_2a_1a_2 & C_1 = a_1a_2a_3a_2a_1a_2a_3 \\
A_2 = a_1a_3a_1a_2a_1a_3a_1 & B_2 = a_1a_3a_1a_2a_3a_1a_3 & C_2 = a_1a_3a_2a_3a_1a_3a_2 \\
A_3 = a_2a_3a_2a_1a_2a_3a_2 & B_3 = a_2a_3a_2a_1a_3a_2a_3 & C_3 = a_2a_3a_1a_3a_2a_3a_1 \\
A_4 = a_2a_1a_2a_3a_2a_1a_2 & B_4 = a_2a_1a_2a_3a_1a_2a_1 & C_4 = a_2a_1a_3a_1a_2a_1a_3 \\
A_5 = a_3a_1a_3a_2a_3a_1a_3 & B_5 = a_3a_1a_3a_2a_1a_3a_1 & C_5 = a_3a_1a_2a_1a_3a_1a_2 \\
A_6 = a_3a_2a_3a_1a_3a_2a_3 & B_6 = a_3a_2a_3a_1a_2a_3a_2 & C_6 = a_3a_2a_1a_2a_3a_2a_1
\end{array}$$

By use of relation (2) these give the 54 independent sequences $S(4)$:

$$\begin{array}{lll}
A_1a_4A_i, & A_1a_4B_i, & A_1a_4C_i, \\
B_1a_4A_i, & B_1a_4B_i, & B_1a_4C_i, \\
C_1a_4A_i, & C_1a_4B_i, & C_1a_4C_i,
\end{array} \tag{6}$$

where $i=1, \cdots, 6$.

It is easily shown that combinations like $A_iA_4A_j$, or $A_iA_4B_j$, *etc.*, or $B_iA_4A_j$, *etc.*, or $C_iA_4A_j$, *etc.*, will not give any independent sequences in addition to those of (6).

We obtain, in this manner, 54 independent solutions to the problem. Clearly these do not include all the required solutions to the problem, because a C sequence corresponding to an S sequence has its first term consist of a single letter, and there exist C sequences whose first terms contain more than a single letter. Also, for that matter, our set of 54 S sequences does not even contain all independent S sequences. The writer has discovered a transformation which, when applied to some of the S sequences in (6), yields new independent sequences.

Editorial Note. The enumeration of permissible C sequences, particularly for the general case of n letters, seems to be difficult, and further communications will be welcome. As a somewhat similar, and perhaps equally difficult, enumeration problem we might here mention the following: Let P_1 be a permutation of n objects. In how many ways can we form a sequence of $n!$ distinct permutations, $P_1, P_2, \cdots, P_{n!}$, such that, for $1 \leq j < n!$, P_{j+1} is obtained from P_j by a single transposition?

Definition of Principal Part

E 846 [1949, 31]. *Proposed by H. J. Hamilton, Pomona College*

The following is typical of many characterizations of the principal part of an infinitesimal which are to be found in elementary calculus texts.

"If an infinitesimal consists of two or more terms of different orders, the term of lowest order is called the *principal part* of the infinitesimal."

Show that this is not definitive and give a valid definition.

Solution by the Proposer. Let α be an infinitesimal and put

$$\beta = \alpha + (\alpha^2 + \alpha^3) = (\alpha + \alpha^2) + \alpha^3.$$

Then, according to the "definition" in the problem, the principal part of β is both α and $(\alpha+\alpha^2)$, each being read off from the corresponding expression above.

A somewhat more subtle example is

$$\beta = \alpha + \alpha^2 - 2\alpha \sin^2 \alpha = \alpha \cos 2\alpha + \alpha^3,$$

in which both α and $\alpha \cos 2\alpha$ satisfy the faulty definition.

An alternative, definitive, characterization of *principal part* follows. *If α and β are simultaneous infinitesimals and if, for some positive integer n, $\lim_{\alpha \to \infty} (\beta/\alpha^n) \equiv c$ exists, is finite, and is not zero, then $c\alpha^n$ is called the* principal part *of β relative to α.* It may be observed that the quantity $(\beta - c\alpha^n)$ is of higher order than β.

Also solved by Leo Moser.

Area in the Ambiguous Case

E 847 [1949, 179]. Corrected Statement. *Proposed by Albert Newhouse, University of Houston*

Let a, b, A be the given parts of a triangle in the ambiguous case. Show that the area of the triangle is given by

$$K = \tfrac{1}{2}b \sin A\,[b \cos A \pm (a^2 - b^2 \sin^2 A)^{1/2}].$$

Solution by P. W. A. Raine, Newport News High School. Assuming $b \sin A < a < b$, $A < 90°$, let x be the altitude to side c and let y and z be the projections of b and a on side c. Then

$$K = \tfrac{1}{2}x[y \pm z] = \tfrac{1}{2}b \sin A\,[b \cos A \pm (a^2 - b^2 \sin^2 A)^{1/2}].$$

Also solved by R. V. Andree, W. R. Beck, Louis Berkofsky, Fannie Boyce, W. G. Brady, D. H. Browne, W. E. Buker, P. L. Chessin, Monte Dernham, Ragnar Dybvik, M. D. Eulenberg, S. E. Field, H. E. H. Greenleaf, Vern Hoggatt, C. H. Holton, R. T. Hood, Albert Hopkins, B. C. Horne, Jr., M. S. Klamkin, Bill Krause, Sam Kravitz, N. D. Lane, Joseph Langr, H. L. Lee, H. R. Leifer, Roger Lessard, Jere Lundholm, R. V. B. Lynch, D. C. B. Marsh, Alta McColl, Leo Moser, C. S. Ogilvy, Mary Payne, C. F. Pinzka, J. W. Ponds, C. C. Richtmeyer, Azriel Rosenfeld, C. M. Sandwick, W. E. Schmitt, N. C. Scholomiti, R. W. Shoemaker, Kirk Stewart, Adrian Struyk, Kaidy Tan, W. R. Talbot, P. D. Thomas, H. B. Thornton, C. W. Trigg, E. H. Vance, A. A. Vuylsteke, Margaret Willerding, Maud Willey, G. A. Williams, Roscoe Woods and the proposer.

E. P. Starke raised the interesting and allied problem of finding those combinations of sides and angles which determine, uniquely or ambiguously, a quadrilateral.

Numbers Decomposable into Sum of Two Abundant Numbers

E 848 [1949, 31]. *Proposed by Leo Moser, University of Manitoba*

Prove that every integer greater than 10^5 can be expressed as the sum of two abundant numbers.

Solution by the Proposer. It is well known that if $(a, b) = 1$ and $n > ab$, then there are positive integers x and y such that $ax + by = n$ (*cf.* Polya and Szegö, *Aufgaben und Lehrsatze aus der Analysis*, vol. 1, p. 4). It is further clear that if a is abundant then so is ax. Now $88 = (2^3)(11)$ and $945 = (3^3)(5)(7)$ are relatively prime abundant numbers, so that every integer greater than (88)(945) can be expressed in the form $88x + 945y$, thus proving the theorem.

Editorial Note. One naturally wonders what is the largest integer not the sum of two abundant numbers.

The two numbers 88 and 945 are the best possible for the proposer's proof inasmuch as 945 is the first odd abundant number and 88 is the smallest abundant number prime to 945. For the abundant numbers not exceeding 10^4 see Glaisher, *Number-divisor Tables*, vol. 8, British Association Mathematical Tables, Cambridge, 1940.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

Problems for Solution

4352 [1949, 414]. Corrected Statement. *Proposed by Paul Erdös, Syracuse University*

Denote by $f(n; a_1, a_2, \cdots, a_k)$ the number of positive integers $m \leqq n$ which are either divisors or multiples of one of the a's $(1 < a_i \leqq n)$. Prove that

$$f(n; a_1, a_2, \cdots, a_k) \leqq f(n; 2, 3, \cdots, p_k),$$

where $2, 3, \cdots, p_k$ are the first k primes.

4355. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Solve the equation

$$5m^2 + 2m + 1 = n^2$$

in integers and show that, provided $m \equiv -1 \bmod 3$, there exists in every system of numeration of base $B = 3m+1$ at least one pair of perfect squares having the form $aabb = (cc)^2$, $bbaa = (dd)^2$. There are infinitely many such systems of numeration. (See also the Proposer's paper, *Concerning two classes of remarkable perfect square pairs*, in this issue of the MONTHLY.)

4356. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Prove the relations:

(a) $$x^{2n+1} - (x-1)^{2n+1} = \sum_{a=0}^{n} \left[\binom{n+a}{2a+1} + \binom{n+1+a}{2a+1} \right] (x^2 - x)^{n-a}$$

(b) $$x^{2n+2} - (x-1)^{2n+2} = (2x-1) \sum_{a=0}^{n} \binom{n+1+a}{2a+1} (x^2 - x)^{n-a}.$$

4357. *Proposed by R. D. Stalley, Stanford University*

Reduce the problem of summing the series

$$\sum_{k=1}^{\infty} k^{-n} x^k,$$

where n is a positive integer $\geqq 2$ to numerical integration of a function over a finite range. (Compare the Proposer's paper, *A Generalization of the Geometric Series* [1949, 325].)

4358. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots$ be a sequence of integers with the property that there does not exist an infinite subsequence of the a's in which no one divides another. Prove that the products

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \qquad 0 \leqq \alpha_i$$

have the same property.

If the a's are chosen to be the first k primes, then we obtain a well-known result due to Dickson.

4359. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If in a tetrahedron one draws lines through the vertices parallel to a given direction Δ and then locates the homothetics of the intersections of these lines with the circumsphere with respect to the centroids of the corresponding faces, in the ratio $-\frac{1}{2}$, the four points so obtained lie in a plane perpendicular to Δ and passing through the Monge point of the tetrahedron. (Compare problem 4233 whose solution appears in this issue of the MONTHLY.)

Solutions

Second Point of Lemoine

4204 [1946, 278]. Corrected Statement. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

In a tetrahedron $T \equiv ABCD$, the tangents of the half-angles $\alpha, \beta, \gamma, \delta$ at the vertices A, B, C, D of the cones inscribed in the trihedral angles (A), (B), (C), (D), are proportional to the barycentric coördinates of the second point of Lemoine of the tetrahedron $T' \equiv A'B'C'D'$ having for vertices the points of contact of the inscribed sphere with the faces BCD, CDA, DAB, ABC.

*Solution by the Proposer.** We shall designate by a, a', b, b', c, c', the dihedral angles of edges BC, DA, CA, DB, AB, DC as well as the lengths of these edges; by A, B, C, D the areas of the faces BCD, CDA, DAB, ABC; and by V the volume of T.

1. We first prove the theorem: The barycentric coördinates of the second point of Lemoine L of a tetrahedron (with this tetrahedron as a reference tetrahedron) are inversely proportional to the normal coördinates of L with respect to the tangential tetrahedron.†

Consider the tetrahedron $T \equiv ABCD$, its circumsphere (O, R), the antiparallel section $A''B''C''$ in the trihedral angle (D) which contains L, and the point D_1 in which DL meets (O, R) again.

$$DL \cdot DD_1 = 2Rt = a'b'c'/k$$

* Translated by W. E. Byrne, Virginia Military Institute.

† V. Thébault, Bull. de la Soc. math. de France, 1948, p. 101.

where k is a proportionality factor,§ and x, y, z, t are the distances from L to the faces of the tangential tetrahedron of T. We have therefore

$$ab'c'x = a'bc'y = a'b'cz = abct.$$

As the normal coördinates of L with respect to T are proportional to (R_1, R_2, R_3, R_4), where R_1 is the radius of the circumcircle of triangle BCD, etc., the barycentric coördinates of L with respect to T are proportional to $(ab'c', a'bc', a'b'c, abc)$, and the theorem follows.

2. If we apply this theorem to $T' \equiv A'B'C'D'$, the normal coördinates (x, y, z, t) with respect to T of the second Lemoine point L' of T' are proportional to

$$\cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c', \qquad \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c',$$
$$\cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c, \qquad \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b \cos \tfrac{1}{2}c.$$

3. If r is the radius of the circumsphere of $A'B'C'D'$, we obtain‖

$$\tan \alpha = r/AB' = \sin \tfrac{1}{2}b \sin \tfrac{1}{2}c \sin \sphericalangle CAB/\cos \tfrac{1}{2}a'.$$

Furthermore¶

$$V = Dh_d/3, \qquad D = \tfrac{1}{2}bc \sin \sphericalangle CAB, \qquad \sin b = bh_bh_d/6V = 3bV/2BD.$$

Hence $\sin \sphericalangle CAB = 2D/bc = 9V^2/2B \cdot C \cdot D \cdot \sin b \sin c$, and

$$\tan \alpha = \frac{9V^2A \cos \tfrac{1}{2}a \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c'}{8A \cdot B \cdot C \cdot D \cos \tfrac{1}{2}a \cos \tfrac{1}{2}a' \cos \tfrac{1}{2}b \cos \tfrac{1}{2}b' \cos \tfrac{1}{2}c \cos \tfrac{1}{2}c'}$$

and

$$\frac{Ax}{\tan \alpha} = \frac{By}{\tan \beta} = \frac{Cz}{\tan \gamma} = \frac{Dt}{\tan \delta}.$$

Note. We obtain thus a complete analogy between the triangle and the tetrahedron. The point L' could be called the Gergonne point of T. R. Bouvaist has indicated other interesting properties of L' in *Mathesis*, Supplément, t. 54, p. 17.

Homothetic Tetrahedrons

4233 [1947, 49]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Parallel lines, of arbitrary direction, through the vertices A', B', C', D' of a tetrahedron $A'B'C'D'$ intersect in A_1, B_1, C_1, D_1 the faces BCD, CDA, DAB, ABC of a homothetic tetrahedron $ABCD$. If k is the homothetic ratio, V the volume of $ABCD$, and V' that of $A_1B_1C_1D_1$, then

$$V_1 = -k^2(2k + 1)V.$$

§ R. Bouvaist, Mathesis, t. 55, 1945–46, p. 352.

‖ Weber-Wellstein, Enzyklopädie der Elementar-Mathematik, Bd. II, 3 Auf., p. 429.

¶ N. A. Court, Modern Solid Geometry, p. 88.

*Solution by the Proposer.** We choose the homothetic center O of the tetrahedrons $T \equiv ABCD$, $T' \equiv A'B'C'D'$ as the origin of a system of rectangular coordinate axes $Oxyz$ with Oz parallel to the direction d of the parallel lines in question. Let x_i, y_i, z_i, $(i=1, 2, 3, 4)$, designate the coördinates of A, B, C, D and kx_i, ky_i, kz_i those of A', B', C', D'. If V is the volume (with sign) of the tetrahedron T, we have $6V = |x_1y_1z_11| = X_1x_1 + Y_1y_1 + Z_1z_1 + T_1$, where $|X_1Y_1Z_1T_1|$ is the adjoint determinant of $|x_1y_1z_11|$. The equation of the plane BCD is $X_1x + Y_1y + z_1z + T_1 = 0$. The coördinates of A_1 are $kx_1, ky_1, -(kX_1x_1 + kY_1y_1 + T_1)/Z_1$. It follows that $6V_1 = -k^2 \left| Z_1x_1 \;\; Z_1y_1 \;\; kX_1x_1 + kY_1y_1 + T_1 \;\; Z_1 \right| / Z_1Z_2Z_3Z_4 = -k^2(k\sum X_ix_i + k\sum Y_iy_i + \sum T_i) = -k^2(12kV + 6V)$. Hence

$$V_1 = -k^2(2k+1)V. \tag{1}$$

As $k^3 = V'/V$, elimination of k from (1) gives

$$V_1 = -\sqrt[3]{V'^2}(\sqrt[3]{V} + 2\sqrt[3]{V'}). \tag{2}$$

It is assumed in the calculations that the direction d is not parallel to the faces of T.

COROLLARY 1. *Parallel lines of arbitrary direction drawn through the vertices of a tetrahedron T meet the opposite faces in A_1, B_1, C_1, D_1. The volume of tetrahedron $A_1B_1C_1D_1$ is three times that of T and of opposite orientation.*[1] *Here $A' \equiv A$, $B' \equiv B$, $C' \equiv C$, $D' \equiv D$, $k=1$, and $V_1 = -3V$.*

COROLLARY 2. *Parallel lines of arbitrary direction drawn through the vertices A', B', C', D' of the tetrahedron T' meet the corresponding faces of the tetrahedron T inversely equal to T' in A_1, B_1, C_1, D_1. The volumes of T and $A_1B_1C_1D_1$ are equivalent.*[2] *In fact, $k=-1$ and $V_1 = V$.*

COROLLARY 3. *If the homothetic ratio $k = -\frac{1}{2}$, A_1, B_1, C_1, D_1 are coplanar since $V_1 = 0$.*

Note. In *Mathesis*, t. 55, p. 380, there appears the statement of a generalization by René Blanchard (Le Havre, France) of our question 3231, as follows: Parallel lines, of arbitrary direction d, drawn through the vertices A', B', C' of a triangle $A'B'C'$ meet the sides BC, CA, AB of a homothetic triangle ABC in A_1, B_1, C_1. If k is the homothetic ratio, S the area (with sign) of ABC and S_1 the area of $A_1B_1C_1$, then $S_1 = -k(k+1)S$. Our problem 4233 is a generalization of Blanchard's problem.

Rotating Elastic String

4282 [1948, 100]. *Proposed by F. W. Herlihy, Comstock, Michigan*

An elastic string (modulus λ, mass ma, unstretched length a) is confined within a straight tube to one end of which it is fastened. The tube rotates around

* Translated by W. E. Byrne, Virginia Military Academy.

[1] Malet, Mathesis, 1885, p. 94; 1890, p. 253.

[2] V. Thébault, Mathesis, t. 55, question 3272, p. 139.

that end with uniform angular velocity ω in a horizontal plane. Find the length of the string in equilibrium.

Solution by Wong Foh Pao, La Universitato Utopia, Shanghai, China. Suppose the tube is perfectly smooth and so narrow that it can just contain the string.

Let ds be the element of the string after stretching and ρ be the density of ds, and let ds_0 be the original length of ds. (s is measured from the fixed end.) Because the mass is unaltered by stretching,

$$\rho ds = m ds_0, \tag{1}$$

and from Hooke's law,

$$\lambda(ds - ds_0)/ds_0 = T,$$

where T is the tension of the string at the position of ds in equilibrium. Since the string is in equilibrium the resultant force acting on ds must be zero,

$$dT/ds = -\rho s\omega^2. \tag{3}$$

Elimination of ρ and s_0 between (1), (2) and (3) gives

$$\frac{dT}{ds} = -\frac{ms\omega^2}{1 + T/\lambda}. \tag{4}$$

An easy integration gives

$$2T + T^2/\lambda = -ms^2\omega^2 + 2H + H^2/\lambda, \tag{5}$$

where the constant of integration is so chosen that $T = H$ when $s = 0$.

s and ρ can be eliminated between (1), (2), (3) and (5) with the result

$$\frac{dT}{ds_0} = -\frac{\sqrt{m}\,\omega}{\sqrt{\lambda}}\sqrt{2\lambda(H - T) + H^2 - T^2},$$

whereupon integration gives

$$\cos^{-1}\frac{\lambda + T}{\lambda + H} = \frac{\sqrt{m}\,\omega}{\sqrt{\lambda}}\, s_0. \tag{6}$$

Here the constant of integration has been chosen so that $T = H$, $s_0 = 0$.

If we assume the length of the string after stretching is L, and put $s = L$ and $T = 0$ (that is at the free end of the string), then (5) becomes

$$mL^2\omega^2 = 2H + H^2/\lambda. \tag{7}$$

In (6) put $s_0 = a$, $T = 0$ (at the free end) to get

$$\frac{\lambda}{\lambda + H} = \cos\frac{a\sqrt{m}\,\omega}{\sqrt{\lambda}}. \tag{8}$$

Finally, elimination of H between (7) and (8) gives the desired result

$$L = \frac{1}{\omega}\sqrt{\frac{\lambda}{m}}\tan\frac{\sqrt{m}\,\omega a}{\sqrt{\lambda}}.$$

Note that ω cannot exceed $\sqrt{\lambda}\,\pi/2a\sqrt{m}$. From (8), if $\omega = \sqrt{\lambda}\,\pi/2a\sqrt{m}$, since $\lambda \neq 0$, $H = \infty$ and from (5) we know that the maximum value of T is H; hence breakage must occur at the fixed end before ω reaches the value $\sqrt{\lambda}\,\pi/2a\sqrt{m}$.

Also solved by I. E. Highberg, Mary H. Payne, F. Underwood, and the Proposer.

Curves Along Which $\bar{z}$ Coincides With An Analytic Function

4283 [1948, 100]. *Proposed by E. P. Starke, Rutgers University*

The conjugate $\bar{z}$ of z, considered as a function of z, is nowhere analytic. Nevertheless, if C is an arbitrary circle or line, there exists a function $f(z)$ such that at every finite point of C, $f(z)$ is analytic and equal to $\bar{z}$. Consider also other curves for which a function exists having the same property.

Solution by Fritz Herzog, Michigan State College. If C is the circle with center at any point a and with radius r, let $f(z) = \bar{a} + r^2/(z-a)$. Then for $|z-a| = r$ we have $f(z) = \bar{a} + \overline{z-a} = \bar{z}$.

If C is the line having slope $\tan\alpha\,(0 < \alpha < \pi)$ and passing through the real point x_0, let $f(z) = x_0 + e^{-2i\alpha}(z-x_0)$. Then for $z = x_0 + e^{i\alpha}t$, with real t, we have $f(z) = x_0 + e^{-i\alpha}t = \bar{z}$.

If C is the line parallel to the real axis and passing through the point iy_0 (with real y_0) let $f(z) = z - 2iy_0$. Then for $z = x + iy_0$, with real x, we have $f(z) = x - iy_0 = \bar{z}$.

In general, let C be any analytic arc, i.e., let C be the mapping of a (finite or infinite) interval $[\alpha, \beta]$ of the real axis by means of an analytic function $\phi(z)$, which is assumed to be schlicht in a sufficiently small (complex) neighborhood of $[\alpha, \beta]$. Then the function $\psi(z) = \overline{\phi(\bar{z})}$ is analytic in a neighborhood of $[\alpha, \beta]$ and maps $[\alpha, \beta]$ on $\overline{C}$, the reflection of C about the real axis. Let $\phi^{-1}(z)$ be the inverse function of $\phi(z)$ and put $f(z) = \psi(\phi^{-1}(z))$. Then $f(z) = \bar{z}$ for z on C.

As an example, consider the parabola whose vertex is at $z = -1$ and whose focus is at $z = 0$. (Every other parabola in the plane can be mapped into the given one by a linear mapping $Az + B$.) In this case, in the above notation, $\phi(z) = (z+i)^2$, and the above procedure leads to the function $f(z) = (z^{1/2} - 2i)^2$, which is single-valued and analytic in a sufficiently small neighborhood of the given parabola; $z^{1/2}$ is to be chosen as $+i$ at $z = -1$.

Editorial Note. If the analytic arc C is given parametrically by $x = x(t)$, $y = y(t)$, a differential equation for the determination of $f(z)$ is easy to obtain. The desired function $f(z)$ is to be such that

$$f(z) = x(t) - iy(t) \tag{1}$$

whenever $z = x + iy$ is a point on C. Furthermore

$$f'(z) = \frac{x'(t) - iy'(t)}{x'(t) + iy'(t)} \tag{2}$$

holds true so long as z remains on C. The required differential equation results from the elimination of t between (1) and (2).

Euler's Constant

4286 [1948, 165]. *Proposed by H. F. Sandham, Trinity College, Dublin, Ireland.*

Prove that

$$\int_0^\infty \frac{\cos x^2 - \cos x}{x}\,dx = \tfrac{1}{2}\gamma,$$

where γ is Euler's constant.

Solution by M. S. Klamkin, Brooklyn Polytechnic Institute, Brooklyn, N. Y. Starting with the known relation for γ,

$$\gamma = \int_0^1 (1 - \cos y)\,\frac{dy}{y} - \int_1^\infty \cos y\,\frac{dy}{y}$$

(see Franklin, *Treatise on Advanced Calculus*, p. 571, eq. 174) we get

$$\left(\frac{1}{r} - \frac{1}{s}\right)\gamma = \int_0^1 (1 - \cos y)\,\frac{dy}{ry} - \int_0^1 (1 - \cos y)\,\frac{dy}{sy} - \int_1^\infty \cos y\,\frac{dy}{ry} + \int_1^\infty \cos y\,\frac{dy}{sy}.$$

Let $y = x^r$ in the first and third integrals and $y = x^s$ in the others, $r, s > 0$. Then

$$\frac{r - s}{rs}\,\gamma = \int_0^\infty \frac{\cos x^r - \cos x^s}{x}\,dx.$$

The proposed result is the special case, $r = 2$, $s = 1$.

Also solved by Robert Breusch, Hwang Cheng-Chung, F. J. Duarte, H. E. Fettis, Philip Franklin, R. O. French, W. J. Harrington, Frank Herlihy, J. G. Herriot, Fritz Herzog, S. Katz and A. M. Peiser, W. H. Marlow, Norman Miller, C. D. Olds, J. H. Simester, R. S. Smith, E. Trost, C. B. Walton, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

An Introduction to Mathematics. By A. N. Whitehead. Twelfth Impression; First American Edition. New York, Oxford University Press, 1948. 6+191 pages. $2.00.

The first printing of Whitehead's classic was mentioned briefly in volume 18 (1911) of this MONTHLY. A more adequate review by R. D. Carmichael appears on pages 282–283 of volume 20 (1913). His hope for a wide circulation for the book has been fulfilled, and it has been reprinted many times since then.

Now, thirty-five years later, a new edition has appeared under the editorial supervision of the author's nephew, J. H. C. Whitehead. The text has been completely reset in smaller type, the diagrams have been redrawn and a few minor errors have been corrected. The contents of the original 256 pages have been compressed into 191, but to compensate for this the price has been increased from the original seventy-five cents.

To readers of more modern books on the significance of Mathematics, Whitehead's *Introduction* may seem a bit old-fashioned. This is particularly apparent in its discussion of applications to physical problems. There is no mention of radio or of television or of the atomic bomb! But the discussion of purely mathematical concepts is as effective as it was when the book first appeared.

Changes in this edition have been kept at a minimum. New footnotes on pages 41, 99, and 152 and a new paragraph on page 105 serve to clarify passages which might be obscure to a non-mathematical reader. It is unfortunate that more changes were not made. For example, we are told (page 55) that Georg Cantor is still living, although the editor was careful on page 111 to refer to the "former" London and North-Western Railway.

Numerous misprints serve to confuse an unwary reader. The omission of an entire line at the bottom of page 114 renders an example meaningless. Misprints of varying degrees of seriousness occur on pages 118, 136, 146 and 157. An up-to-date reading list is given in the Bibliography, but here again an error occurs (page 188) in the initials of J. W. Young.

In spite of these minor flaws, we believe (to quote the previous review) that "the book deserves a wide circulation."

H. M. GEHMAN

Mathematics of Finance. By P. M. Hummel and C. L. Seebeck, Jr. New York, McGraw-Hill Book Co., 1948. 7+365 pages. $4.00.

A few months ago the reviewer became involved in an argument with a

non-mathematician concerning some now-forgotten point in practical finance. Calling to mind the reverence of the uninitiated for the printed word, he picked up the present book, which had remained for some time unopened on his shelves, surrounded by its many humble relatives. The argument was quickly settled, of course. At no little surprise to himself, however, the reviewer spent the next hour in reading the Hummel-Seebeck text with genuine interest and appreciation.

The excellent format of the book contributes no little to the good impression. A more important factor, nevertheless, is the clarity and simplicity of the writing. It would seem that the authors conceived of their readers as people of intelligence: too intelligent to be satisfied by confused explanations and intelligent enough to appreciate carefully stated principles. Perhaps the best chapter of the book is Chapter III (Equations of Equivalence) in which, for example, the authors define equivalence of "dated values" and go on to prove transitivity of equivalence. It is also encouraging to note that general annuities are handled by reducing them to simple annuities, thereby keeping the emphasis on general principles and avoiding a whole complex of complicated formulas.

The book seems eminently suited for students with a little knowledge of elementary algebra. It abounds in illustrative examples and problems. There are appendices on abridged multiplication, common logarithms and progressions, as well as fifteen double entry tables. Almost the only error noted occurs on page 203, where two headings in an illustrative table have been interchanged. The following list of chapter headings sufficiently indicates the contents: I. Simple Interest. II. Compound Interest. III. Equation of Equivalence. IV. Simple Annuities. V. Ordinary General Annuities. VI. Perpetuities. VII. Amortization and Sinking Funds. VIII. Bonds. IX. Depreciation. X. General Annuities—Advanced Topics. XI. Approximating Methods. XII. Life Annuities and Life Insurance.

R. H. BRUCK

Analytic Geometry. By P. R. Rider. New York, The Macmillan Co., 1947. 10+383 pages. $3.25.

This text is suitable for a brief or an extended course in analytic geometry. In addition to eleven chapters on the traditional topics, the book contains a chapter on curve fitting, three chapters on space analytic geometry, ten pages of tables, an adequate index, and answers to the odd numbered problems.

As might be expected from this author, the exposition is carefully made and well-motivated. To this reviewer the arrangement of topics seems quite satisfactory. The lists of exercises are more useful than in the author's well known algebras, for here there are long lists of the simpler problems and only a reasonable number of the more difficult problems. Since, on the whole, the 203 figures are excellently done and will, by themselves, arouse considerable student interest, it is unfortunate that some of these contain misleading errors, such as Figures 167 and 170. The illustrative examples well merit the author's claim

that they have been chosen with extreme care to contribute as much as possible to student understanding of the principles of the corresponding sections.

In summary, this Rider text certainly deserves consideration by those who are looking for a fresh, adequate, teachable treatment of analytic geometry.

B. M. STEWART

Analytic Geometry. By P. K. Rees and E. D. Mouzon. New York, The Dryden Press, 1948. 28+305 pages. $2.75.

This is a textbook presenting the standard topics of analytic geometry. It is well arranged, quite legible, and its figures are quite clear.

At the beginning of the text there is included a review of the algebra, trigonometry, and geometry necessary as a background. This material is arranged according to subject matter, and again according to the article in which it is used. In the second listing the number of the article appears before the material making it easy to use for reference purposes.

The central conics are treated in parallel columns on the same pages. In this manner it is easier to note the similarities between them. The problem of determining the points of intersection of two curves when their equations are given in polar form is treated much more exhaustively than in most texts. The work on loci and equations is introduced at an early stage and is carried out in good detail. In this connection it is felt that a more thorough treatment might have been given to the sketching of the graphs of polar equations. For example, no mention is made of the methods of finding tangents at the pole, and the polar equations of the conics are assigned as a single problem without discussion.

Two chapters on solid analytic geometry are included in the text. These chapters provide an introduction to the subject and cover planes, lines, and quadric surfaces.

This text is quite suitable for the average student who requires a background in the fundamentals of analytic geometry rather than the student who would like to delve more deeply into the subject. The review at the beginning of the text is quite helpful, although a mention of radian measure might have been made. Also it is felt that some material on the sketching of surfaces other than the quadrics, and of solids bounded by two or more surfaces would have added to the usefulness of the text.

Few inaccuracies occur in the text. The most conspicuous error is in the definition of logarithms given on page 168.

J. H. BELL

Solid Geometry. By J. S. Frame. New York, McGraw-Hill Book Co., 1948. 9+339 pages. $3.50.

While this text can be used profitably in high schools, it is also a book upon which a course in solid geometry of undeniably college caliber can be based. In fact, because of the more than generous list of topics covered and their convenient arrangement, any one of several courses can be given with this

book as text. If the instructor prefers, the course can be handled in the more or less traditional fashion as a succession of definitions and theorems and proofs of theorems: the text is admirable for this purpose in that it furnishes adequate examples but leaves many of the proofs as exercises. It is possible, also, to de-emphasize theorem-proving and to give more attention to solid mensuration and to the drawing of two-dimensional representations of three-dimensional figures. A feature of the book is the emphasis placed on the last-mentioned topic. Methods and aids are presented throughout Parts I, II and III, and Part IV is devoted exclusively to projections and maps. A patented device known as the trimetric ruler is provided with the text to facilitate the construction of drawings from which three-dimensional figures can be readily visualized. By completing the text in fifty college class hours an instructor should be able to present a course which will train the student to make careful proofs of theorems, equip him with detailed knowledge of solid mensuration, and give him far greater competence in drawing than is usually the case.

Particularly gratifying are the author's definitions, which are frequently "tighter" than those usually found. Examples are his "collinear," "line segment," "congruent figures," "straight line," "on the same side of," "in the same direction," and "plane." Praiseworthy, too, is his employment of certain expressions which facilitate the statement of definitions and theorems: for example, "mediator of a line segment" takes the place of "plane which bisects and is perpendicular to the line segment."

One might wish that the author, in his laudable attempt to decrease the ambiguity resulting from the multiple meanings of such expressions as "dihedral angle," had not used the words "dihedron" and "trihedron," since their most readily suggested generalizations clash with the traditional meaning of "polyhedron." Also, two definitions of "plane perpendicular to plane" are given, one on page 36 and the second on page 51, their equivalence being stated as a theorem whose proof is set as an exercise. The obvious suggestion is that one be stated as a definition and the other as a theorem. Some might wish, also, that the appearance of Cavalieri's Theorem had been postponed as in traditional texts until after a few volume proofs had been given without it or with proven special cases of it.

The book is singularly free of typographical errors, and careful attention has been paid to grammar.

This reviewer is most happy that this text has appeared. He has used it as a reference work in a college course in solid geometry, and recommends it highly for its rigor, the breadth and detail of its coverage, the convenient arrangement of its topics, and its generous lists of well-chosen exercises.

L. D. RODABAUGH

NEW BOOKS RECEIVED

Arithmetic for Teacher-Training Classes. 3rd Edition. By E. H. Taylor and C. N. Mills. New York, Holt, 1949. 6+441 pp. $3.00.

Smithsonian Elliptic Functions Tables. Vol. 109. By G. W. Spenceley and R. M. Spenceley. Washington, D. C., Smithsonian Institution, 1947. 4+366 pp.

Freshman Mathematics. 3rd Edition. Revised by C. V. Newsom. New York, Rinehart, 1949. 16+559 pp. $5.00.

Numbers We See. By A. Riess, M. L. Hartung, and C. Mahoney. New York, Scott, Foresman, 1948. 162 p. $1.32.

Your Mathematics. By G. E. Hawkins and G. Tate. New York, Scott, Foresman, 1948. 592 pp. $2.20.

Concepts of the Calculus. By C. B. Boyer. Reprint, Foreword by R. Courant. New York, Hafner, 1949. 12+346 pp. $5.50.

The Mathematical Analysis of Logic. By George Boole. Reprint. New York, Philosophical Library, 1948. 6+82 pp. $3.75.

On the Theory of Stochastic Processes and Their Application to the Theory of Cosmic Radiation. By N. Arley. New York, Wiley, 1949. 240 pp. $5.00.

Klassische Funktionentheorie. Vol. 2. By H. Behnke and F. Sommer. Münster, Aschendorffsche, 1948. 8+234 pp. DM 6—.

Introduction to Analytic Geometry and the Calculus. By H. M. Dadourian. New York, Ronald Press, 1949. 10+246 pp. $3.25.

Five-figure Tables of Mathematical Functions. 2nd Edition. By J. B. Dale. London, Arnold, 1949. 8+121 pp. $1.50.

The Life and Works of Herbert Ellsworth Slaught. By H. J. Dark. Nashville, George Peabody College for Teachers, 1948. 8+152 pp.

Kurvenintegrale und Begründung der Funktionentheorie. By L. Heffter. Berlin, Springer, 1948. 4+48 pp. DM 5.40.

Rank Correlation Methods. By M. G. Kendall. London, Griffin, 1948. 8+160 pp. 18s.

Analytic Geometry. By A. L. Nelson, K. W. Folley and W. M. Borgman. New York, Ronald Press, 1949. 8+215 pp. $3.00.

Fehlertheorie und Ausgleichung von Rautenketten in der Nadirtriangulation. (Veröffentlichungen de Geodätischen Institutes in Potsdam, no. 1). By K. Reicheneder. Berlin, Akademie, 1949. 8+98 pp. DM 11—.

Vektorrechnung. Vol. 2. By F. K. Schmidt. Münster, Aschendorffsche, 1948. 8+244 pp. DM 6.40.

Calculus. By L. L. Smail. New York, Appleton-Century Crofts, 1949. 16+592 pp. $4.50.

Solid Analytic Geometry. By A. Albert. New York, McGraw-Hill, 1949. 10+162 pp. $3.00.

Commercial Algebra. By C. Bell and L. J. Adams. New York, Holt, 1949. 8+304 pp. $2.75.

Mathematics of Finance. By C. Bell and L. J. Adams. New York, Holt, 1949. 8+366 pp. $2.75.

You Can't Win. By E. E. Blanche. Washington, Public Affairs Press, 1949. 155 pp. $2.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

KAPPA MU EPSILON CONVENTION

The national mathematics fraternity, *Kappa Mu Epsilon*, held its seventh Biennial Convention at Topeka, Kansas on April 10–12, 1949 with *Kansas Delta* chapter of Washburn Municipal University acting as host. There were 184 delegates in attendance representing 35 chapters.

Addresses were given by Dr. L. M. Graves, University of Chicago, who spoke on *The role of generalization and abstraction in mathematics*, and Dr. G. B. Price, University of Kansas, who spoke on *Some famous problems of modern mathematics*. Papers were also presented by nine student members, while eleven other student papers were presented by title.

Two new chapters, Central College of Fayette, Missouri, and Mississippi Southern College of Hattiesburg, Mississippi were admitted to the fraternity. This brings the total number of chapters to forty-two.

National officers elected for 1949–51 were: President, H. Van Engen, Iowa State Teachers College; Vice-President, H. D. Larsen, Albion College; Secretary, E. Marie Hove, Hofstra College; Treasurer, L. F. Ollmann, Hofstra College; Historian, C. C. Richtmeyer, Central Michigan College. Dr. H. D. Larsen was reappointed to the editorship of the *Kappa Mu Epsilon* publication—*The Pentagon*.

CLUB REPORTS, 1947–48

Kappa Mu Epsilon, Northeastern State College

Titles of papers read at meetings of the *Oklahoma Alpha* Chapter of *Kappa Mu Epsilon* include:

Astronomy, by Dean L. P. Woods

Solution of higher equations, by James Barringer

Teaching secondary mathematics, by Prof. Vella Frazee

Discussion and solution of determinants, by Charles Brown and Thomas Summers

Exponential functions, e, by J. B. Willis

Moments of magic, by Prof. Ross Anderson, who spoke at the annual Founder's Banquet.

The officers for 1947–48 were: President, Robert Johnston; Vice-President Kermit Stuart; Treasurer, William Spicer; Secretary, Nellie Elledge.

Mathematics Club, Illinois Institute of Technology

The *Mathematics Club* of IIT reports a larger interest and attendance at the meetings during the past year during which the following papers were

presented:

Solution of equations by successive approximations, by Prof. L. R. Ford
Jordan's theorem, by Prof. Karl Menger
Dynamic families, by Dr. John De Cicco
The concept of mathematical limit, by Dr. LeRoy Wilcox
Lie theory of differential equations, by Richard Edwards
The real number system, by David Rubinfien
Group theory, by Donald Friedlen and Bob Natkin
Boolean algebra, by Marshall Kaplan and Donald Friedlen
Non-Euclidean geometry, by Malcolm Smith.

The officers are: President, Malcolm Smith; Program Chairman, Marshall Kaplan.

Mathematics Club, Case Institute of Technology

The activities of the *Case Mathematics Club* during 1947–48 included the following talks given by members:

A simple application of integral equations, by Prof. S. W. McCuskey
Some topics in number theory, By Ernest Leach
Euclid's parallel postulate, by J. E. Darraugh
Fourier series and boundary value problems, by Ernest Leach
The Pythagorean theorem, solved in integers, by Peter Stephen
A problem in differential geometry arising in the theory of radar antennae. by Prof. R. F. Rinehart.

Officers were: President, Ernest Leach; Faculty Adviser, Prof. Max Morris.

Kappa Mu Epsilon, Albion College

The *Michigan Alpha* Chapter of *Kappa Mu Epsilon* reports the following papers presented during 1947–48:

Galileo, by William Pillinger
Self-taught mathematicians, by Edward Eames
Descartes, by William Beers
Arithmetic revisited, by Prof. H. D. Larsen
How the middle ages counted, by Charles Bishop
4000 years for numerals, by William Doddrill
Evolution of our exponential notation, by Raymond Gillespie
The zero, by Lucy Richardson
Fallacies in mathematics, by Dorothy Manley.

A recognition evening for Prof. E. R. Sleight, who retired in June after 41 years at Albion College, as well as a joint picnic with the Chemistry and Physics Clubs were held.

The officers elected for 1948–49 are: President, William Schofield; Vice-President, Eugene Snell; Secretary-Treasurer, Barbara Barnes; Faculty Sponsor, Prof. H. D. Larsen.

Mathematics Club, Hunter College

The *Mathematics Club* of Hunter College had a most successful and enjoyable year with numerous well attended program meetings and social functions. The student speakers and their topics were:

Quaternions, by Marion Boykan
Mathematics and nature, by Cecile Cohen
Dedekind's theory of numbers, by Luisa Maissonet
Non-Euclidean geometry, by Ruth Friedman
Mathematics and music, by Minna Gottlieb
Finite geometry, by Elaine Traub
History of probability, by Evelyn Margoe
Cryptoanalysis, by Anita Fernbach and Florence Berg.

Faculty and guest speakers and their papers were:

Elliptic functions, Prof. J. H. Bushey
Modern ciphers, by Prof. L. S. Hill
Some unsolved problems of number theory, by Dr. Mary Dolciani, Research Scholar at the Institute for Advanced Study.

The social activities consisted of a winter and a spring dinner, a Christmas party, a theatre party, and a boat ride to Bear Mountain.

The officers of the club were: President, Marcia Geiger; Vice-President, Martha Friedler; Secretary, Luisa Maissonet; Treasurers, Ethel Diamont and Florence Miroff.

Mathematics Club, Adelphi College

In addition to the regular meetings, the *Mathematics Club* of Adelphi was privileged to hear the following guest speakers:

The teaching of mathematics and its applications, by Dr. C. N. Shuster, State Teachers College, Trenton, New Jersey

Pleasures in mathematics, by Prof. J. Ginsberg, Chairman of the Mathematics Department of Yeshiva College.

Trips were taken by the members to Brooklyn, New York, to visit the Electronized Chemical Corporation, and to Bethpage, New York, to visit the Gruman Aircraft Engineering Corporation.

Kappa Mu Epsilon, Drake University

The *Iowa Beta* Chapter of *Kappa Mu Epsilon* reports the following programs given during 1947–48:

Opportunities for students in mathematics, by Walter Potts, Jr.

The field of engineering, by Everett Gilman, a talk supplemented by two movies "Engineering" and "Electric Currents"

History and construction of magic squares, by William Chappell.

Informal social and initiation meetings were also held.

Officers serving during the year were: President, Robert Yohe; Vice-President, Walter Potts, Jr.; Secretary, Dean Williams; Treasurer, Robert Barkus; Faculty Sponsor, Prof. E. L. Canfield.

Junior Mathematical Club, University of Chicago

Meetings were held approximately every two weeks during the Autumn, Winter, and Spring quarters, alternating with the meetings of the *Senior Mathematical Club*. Papers presented to the *Junior Club* are intended primarily to be of interest to students of about beginning graduate level. The following papers were presented:

Transfinite numbers, by Prof. R. W. Barnard

Algebraic number fields, by Harley Flanders

Moral expectation, by Dr. L. J. Savage

Unique connected graphs, by Prof. J. K. Senior, Department of Chemistry

Foundations for a theory of genealogical systems, by Prof. L.R. Willcox, Illinois Institute of Technology

On some numerical methods, by Efrem H. Ostrow

Lie algebras and quantum kinematics, by Prof. I. E. Segal

Bernstein polynomials and their derivatives, by George Klein

Linear graphs and the theory of transportation, by Prof. T. C. Koopmans, Department of Economics

Continued fractions and the Stieltjes integral, by J. V. Finch

Ideals and topologies, by Dr. Edwin Hewitt

Real fields, by Isador Singer

The spectral theorem, by Prof. M. H. Stone.

At the final meeting of the year it was announced that Isador M. Singer was this year's recipient of the annual prize awarded for a paper presented by a student.

Officers for 1948–49 are: President, I. M. Singer; Treasurer, C. E. Miller; Faculty Adviser, Prof. M. H. Stone.

Mathematics Club, Regis College

Among the interesting activities of the *Mathematics Club* of Regis College were the following addresses:

Mathematics in art, by Edna Cunningham '48

What should be the content of the course in mathematics required of the graduate of a liberal arts college?, by Sister M. Leonarda

Things you have not noticed, by Barbara Lane

Harmonic motion, by Ruth P. Carell

The nine-point circle, by Phyllis Moran

Formulas for the area of a triangle, by Elinor O'Neil

A biographical sketch of Colin Maclaurin, by Katharine Healy

The map-maker's proposition, by Anne McDonnell

The evaluation of pi, by Mary T. Harrington

The geometries of Lobachevski and Riemann, by Marie Madden

Chance and chanceability, by Ann McCarthy

The d'Alembert, Euler, Bernoulli controversy, by Barbara A. Sullivan

A simplification of the second derivative test, by Anna McFarlane
Integration of parts, by Mariam Brault
An ornithological note, by Ruth P. Carell
Wheatstone bridges, Maryann Boyce
Seismology, by Phyllis Moran
The atomic bomb, by Catherine T. Walsh
Our hobby—architecture, by Louise Kelley and Barbara Lane
Pascal's triangle and negative exponents, by Anne McDonnell.

Two social events, a field trip, and representation at some New England mathematical meetings represented other activities. Members of the Club presented books to the College Library, wrote mathematical articles for publication in various journals, and published several issues of the *Mathematical Angle.*

Officers elected for 1948–49 are: President, Barbara Lane; Vice-President, Katharine Healy; Secretary, Maryann Boyce; Treasurer, Virginia Lee; Moderator, Sister M. Leonarda.

Pi Mu Epsilon, University of Oregon

The annual report of the *Oregon Alpha* chapter of *Pi Mu Epsilon* includes the following list of papers read before the club:

The Canadian Mathematical Congress, by Prof. K. S. Ghent
Proof that a trigonometric function of an angle having an integral number of seconds is an algebraic number, by Walter Gilbert
Correlation theory, by Prof. Fréchet of the University of Paris
Fourier series, by Shirley K. Anderson
Mathematics of life insurance, by Frank Howatt
A generalization of the average of two numbers, by Prof. B. H. Arnold of Oregon State College.

Ten dollars was contributed to the E. E. DeCou mathematics prize. This prize, in honor of the former head of the mathematics department, was started this year and will be given to an outstanding upper division student in mathematics.

The officers elected for 1949–50 are: Director, John E. Olson; Vice-Director, Carl Pride; Secretary-Treasurer, Gene Thompson.

Mathematics Club, McMaster University

The *Mathematics Club* of McMaster University held six regular meetings during the academic year 1947–48. Membership in the Club numbered fifty. Topics presented by members and guests were:

Life of Euler, by Lloyd Rollerson
Spiral nebulae, by Sidney Hillyer
Simple algebraic proof that the limit e exists, by Norman Lang
Calculating machines, by Eric McAllister

Two theorems concerning polyhedra, by Dick Beesack

The structure of the nucleus, by Dr. Martin Johns

Actuarial work, by Mr. D. Campbell of Crown Life Insurance Company

The Cornu spiral with applications to the field of optics, by Miss Catherine Zavitz.

Officers for the year 1948–49 are: President, Robert Mitchell; Vice-President, Eric McAllister; Secretary-Treasurer, Dick Beesack; Social Convenor, Audrey Baker; Year Representatives, Don McTavish, Frances Wardle, and Keith Rosebrugh.

Mathematics Club, University of Colorado

The *Mathematics Club* of the University of Colorado meets each two weeks at which time a topic of mathematical interest is presented. Usually the discussion is on a sufficiently elementary plane so as to be of interest to students having only calculus. Among the topics presented during 1947–48 were:

Selected topics from modern algebra, by Prof. A. B. Farnell

The foci of plane curves, by Prof. Claribel Kendall

Continued fractions, by Burrowes Hunt

Fourier series and the development of mathematics, by H. Bartram

Quaternians, by Gideon Culpepper.

Officers for 1948–49 are: President, Roy F. Reeves; Secretary-Treasurer, David DeVol; Sponsor, Prof. B. W. Jones.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

The fourth Stanford University Competitive Examination in Mathematics (see this MONTHLY, vol. LIII, no. 7, pp. 406–409 (1946)) was held April 2, 1949, in 37 high schools in California; 189 students took part. The following problems were proposed:

1. Prove that no number in the sequence

$$11, 111, 1111, 11111, \cdots$$

is the square of an integer.

2. The three sides of a triangle are of lengths, l, m, and n, respectively. The numbers l, m, and n are positive integers,

$$l \leqq m \leqq n.$$

(a) Take $n = 9$ and find the number of different triangles of the described kind.

(b) Take various values of n and find a general law.

3. (a) Prove the following theorem: A point lies inside an equilateral triangle and has the distances x, y, and z from the three sides respectively; h is the altitude of the triangle. Then

$$x + y + z = h.$$

(b) State precisely and prove the analogous theorem in solid geometry concerning the distances of an inner point from the four faces of a regular tetrahedron.

(c) Generalize both theorems so that they should apply to any point in the plane or space, respectively (and not only to points inside the triangle or tetrahedron). Give precise statements and, if you have time, also proofs.

The writer of the best paper, D. E. Johansen, student at Palo Alto High School, Palo Alto, California, received a $500 scholarship at Stanford University. D. B. Toy of Los Angeles, a student at the Harvard School in North Hollywood, California, and Frank Paulsen, a student at Castlemont Senior High School in Oakland, California, received "Honorable Mention."

CORRIGENDA

Professor R. C. Archibald, author of the *Outline of the History of Mathematics*, sixth edition, which was published as the second Slaught Memorial Paper, submits the following list of corrigenda:

Page 6, line -9, *for* 1948, *read* 1949
Page 9, line 15, *for* Pythagoras., *read* Pythagoras:
Page 9, line 23, *for* 650 700, 649 909, 1 008 541, *read* 13 500, 12 709, 18 541
Page 38, line 4, *for* over, *read* nearly
Page 40, line -3, *for* sec $\alpha - \tan \alpha$, *read* sec $\alpha + \tan \alpha$
Page 49, line 11, *for* tan x were, *read* tan x (x rational) were
Page 80, line -3, *for* [59], *read* [69]
Page 101, line 15, *for* M'Cay, Schoute, *read* M'Cay, Neuberg, Schoute
Page 101, line 17, *for* Mandart, Neuberg;, *read* Mandart;
Page 104, line 4, *for* death are, *read* death, are
Page 104, *for* Archibald (1875–), *read* Archibald, R. C. (1875–)
Page 107, col. 1, line -8, *for* 29, *read* 39.

PERSONAL ITEMS

Professor A. W. Boldyreff, University of New Mexico, was the representative of the Association at the inauguration of President T. L. Popejoy of the University of New Mexico on June 4, 1949.

Dean O. H. Rechard of the College of Liberal Arts, University of Wyoming, has been appointed to represent the Association at the celebration of the Seventy-Fifth Anniversary of the founding of the Colorado School of Mines at Golden on September 29, 1949.

Mr. W. D. Lambert, formerly of the U. S. Coast and Geodetic Survey, has been elected to membership in the National Academy of Science. He has also been awarded the William Bowie Medal for leadership in the study of earth sciences by the American Geophysical Union.

Professor Saunders MacLane of the University of Chicago has been elected to membership in the American Philosophical Society and the National Academy of Sciences.

Assistant Professor Wilfred Kaplan of the University of Michigan, Professor S. C. Kleene of the University of Wisconsin, Associate Professor G. W. Mackey of Harvard University, and Associate Professor J. W. Tukey of Princeton University have been awarded Guggenheim Fellowships.

Drs. E. E. Moise and M. A. Woodbury of the University of Michigan have been granted National Research Council Fellowships providing for postdoctoral study in the Institute for Advanced Study.

Cornell University makes the following announcements: The Mathematics Department is expanding research and instruction in the theory of probability and its applications with the continued support of a research contract with the Office of Naval Research; Professors Feller, Kac, Chung and Dr. Donsker are participating in this work. Professor G. Elfving of the University of Helsingfors has been appointed Visiting Professor of Mathematical Statistics for the academic years 1949–51; Professor J. L. Doob, on sabbatical leave from the University of Illinois, will spend the year 1949–50 at Cornell; Dr. Gilbert Hunt has been appointed Assistant Professor of Mathematics.

De Paul University announces the following: Professor Rufus Oldenburger chairman of the Mathematics Department, has been granted a leave of absence to devote his full time to the Woodward Governor Company, where he has been mathematician-engineer since 1942; Associate Professor John De Cicco of the Illinois Institute of Technology has been appointed Visiting Professor and Acting Chairman of the Department of Mathematics for the academic year 1949–50; Dr. C. W. Moran of Wright Junior College and Dr. Jerome Sachs of the Chicago Teachers College have been appointed Lecturers of Mathematics in the graduate school.

Kent State University reports: Assistant Professor C. L. Riggs of the University of Kentucky has been appointed to an assistant professorship; Mr. Myron Cox, Virginia Polytechnic Institute, Mr. Donald Dunning of North

Central College, Mr. D. W. Seibel, Denison University, have been appointed to graduate assistantships.

Michigan State College announces the following promotions and appointments: Associate Professor J. D. Hill has been promoted to a professorship; Assistant Professors J. H. Bell and Leo Katz have been promoted to associate professorships; Mr. J. W. Coy of the University of Michigan has been appointed to an instructorship.

Tarleton State College reports the following: Associate Professor L. G. Worthington has been promoted to the position of Professor and Head of the Department of Mathematics; Professor A. A. McSweeny and Assistant Professor Mary Marrs have been placed upon a status of modified service.

Professor D. B. Ames of Rensselaer Polytechnic Institute has been appointed Chairman of the Department of Mathematics at the University of New Hampshire.

Dr. R. V. Andree of the University of Wisconsin has accepted an assistant professorship at the University of Oklahoma.

Associate Professor W. C. Arnold of De Pauw University has been promoted to a professorship.

Dr. Nachman Aronszajn of the Centre National de la Recherche Scientifique, Paris, has been appointed to a research professorship at Oklahoma Agricultural and Mechanical College.

Professor T. B. Ashcraft, head of the Department of Mathematics of Colby College, has retired with the title of Professor Emeritus.

Professor O. F. H. Bert, Washington and Jefferson College, has retired with the title of Professor Emeritus.

Mr. Joseph Blum is now located at Los Alamos Scientific Laboratory.

Dr. Mary L. Boas has been appointed Lecturer at Wellesley College.

Professor Enrico Bompiani of the University of Rome has been appointed to a professorship at the University of Pittsburgh; he will spend half of each academic year at the University of Rome.

Mr. C. W. Cassel of the University of Dayton has accepted a position as chief of the Branch of Measurement and Analysis, Engineering Division, Clinton County Air Force Base, Wilmington, Ohio.

Mr. R. E. Edwards has been appointed Associate Actuary by the Baltimore Life Insurance Company.

Dr. H. E. Ellingson of Rosemount Research Center, University of Minnesota, has been appointed Mathematician with the Naval Ordnance Laboratory, Washington, D. C.

Dr. A. W. Goodman of Rutgers University has been appointed to an associate professorship at the University of Kentucky.

Miss Virginia M. Hall has been appointed to an instructorship at Simmons College.

Professor E. D. Hellinger of Northwestern University has been appointed Visiting Professor of Mathematics at Illinois Institute of Technology.

Professor W. R. Hutcherson of Northwestern State College has been appointed to a professorship at the University of Florida.

Dr. H. G. Landau of the Ballistic Research Laboratories, Aberdeen Proving Ground, has accepted a position as research associate with the Committee on Mathematical Biology, University of Chicago.

Professor E. H. Larguier, Spring Hill College, was granted a leave of absence for the summer term to accept a position as visiting professor at St. Louis University.

Mr. George Laush, Cornell University, has been appointed to an assistant professorship at the University of Pittsburgh.

Dr. Joseph Lehner, formerly of Hydrocarbon Research, Inc., New York has accepted a position as associate professor at the University of Pennsylvania.

Dr. Samuel Lubkin has been appointed Consultant to the Machine Development Laboratory, Applied Mathematics Laboratories, National Bureau of Standards.

Dr. R. K. Luneberg of the Institute for Mathematics and Mechanics of New York University has been appointed to an associate professorship at the University of Southern California.

Professor H. B. Mann of Ohio State University has been appointed to the Applied Mathematics Laboratories of the National Bureau of Standards.

Associate Professor C. G. Maple of North Texas State College has accepted an associate professorship at Iowa State College.

Professor Dorothy McCoy of Belhaven College has been appointed to a professorship at Wayland College.

Assistant Professor Paul Meier of Lehigh University is now on the staff of the Philadelphia Tuberculosis and Health Association in the position of Research Secretary.

Dr. E. P. Miles, Jr., of Duke University has been appointed to an associate professorship at Alabama Polytechnic Institute.

Rear Admiral R. E. Nelson (U.S.N. retired), professor of naval science at Dartmouth, has accepted a position as associate professor of mathematics at Dickinson College.

Associate Professor B. J. Pettis of Tulane University has been promoted to a professorship.

Mr. D. W. Pounder is now employed as an aerodynamicist by A. V. Roe Canada, Ltd., Toronto.

Assistant Professor F. M. Pulliam of the University of Kentucky has been appointed Assistant Professor of Mathematics and Mechanics at the United States Naval Postgraduate School.

Mr. Gordon Raisbeck is now a member of the staff of the Bell Telephone Laboratories, Murray Hill, New Jersey, in the acoustics division.

Mr. L. L. Rauch, Princeton University, has been appointed to an assistant professorship at the University of Michigan.

Assistant Professor W. P. Reid of Purdue University has accepted an as-

sistant professorship at the Air Forces Institute of Technology, Wright Field, Ohio.

Associate Professor E. K. Ritter of the United States Naval Postgraduate School has accepted a position as supervisor, Performance Analysis Group, Aeronautical Research Center, University of Michigan.

Miss Joan Robinson has been appointed to an instructorship at Wilson College.

Dr. C. E. Sealander of Ohio State University has joined the staff of Battelle Institute, Columbus, where he will be engaged in research in engineering physics.

Assistant Professor E. B. Shanks, Vanderbilt University, has been promoted to an associate professorship.

Professor Emeritus E. W. Sheldon of the University of Alberta has been appointed to an interim professorship at Acadia University.

Professor Otto Szasz of the University of Cincinnati has been granted a year's leave of absence to do research in the National Applied Mathematics Laboratories of the National Bureau of Standards, Los Angeles.

Associate Professor J. R. Vatnsdal of State College of Washington has been promoted to a professorship.

Mr. S. I. Vrooman of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. Daniel Zelinsky of the Institute for Advanced Study has been appointed to an assistant professorship at Northwestern University.

Professor A. Buhl of the University of Toulouse died on March 24, 1949.

Professor Torsten Carleman of the University of Stockholm died on January 11, 1949.

Dr. Aristide Fanti, formerly scientific librarian of the National Bureau of Standards, died on April 5, 1949.

Dr. H. V. Gummere of Philadelphia died on February 9, 1949.

Mr. E. S. Manson, formerly professor of astronomy at Ohio State University, died on January 29, 1949.

Professor Emeritus Helen A. Merrill of Wellesley College died on May 1, 1949.

Professor Emeritus L. R. Perkins of Middlebury College died April 27, 1948. He was a charter member of the Association.

Professor William Threlfall of the University of Heidelberg died on April 4, 1949.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

JOINT MEETING OF THE ASSOCIATION WITH A.S.E.E.

A meeting of the Mathematical Association of America was held at Rensselaer Polytechnic Institute, Troy, New York, on Monday and Tuesday, June 20–21, 1949, in conjunction with the annual meeting of the American Society for Engineering Education. All sessions of the Association were joint sessions with the Mathematics Division of the A.S.E.E.

About one hundred and fifty persons attended the sessions of the Association including the following eighty-eight members of the Association:

E. B. Allen, Rensselaer Polytechnic Institute
N. H. Ball, U. S. Naval Academy
D. H. Ballou, Middlebury College
R. H. Bardell, University of Wisconsin
R. A. Beaver, N. Y. State College for Teachers at Albany
Brother Bernard Alfred, Manhattan College
W. W. Bessell, U. S. Military Academy
J. S. Biggerstaff, Rensselaer Polytechnic Institute
Harry Birchenough, N. Y. State College for Teachers at Albany
I. S. Boak, N. Y. State Agricultural and Technical Institute, Canton
L. F. Boron, University of Maine
N. H. Bryan, Clemson College
F. J. H. Burkett, Union College
E. A. Butler, N. Y. State College for Teachers at Albany
W. E. Byrne, Virginia Military Institute
J. D. Campbell, Rensselaer Polytechnic Institute
J. W. Cell, North Carolina State College
L. H. Chambers, U. S. Naval Academy
W. F. Cheney, Jr., University of Connecticut
W. H. H. Cowles, Pratt Institute
L. J. Deck, Muhlenberg College
Arnold Dresden, Swarthmore College
W. M. Duke, Cornell Aeronautical Laboratory
J. N. Eastham, Cooper Union
Clarence Ford, Male High School, Louisville
L. R. Ford, Illinois Institute of Technology
R. M. Foster, Polytechnic Institute of Brooklyn
A. H. Fox, Union College
J. B. Garrett, Jr., Siena College
H. M. Gehman, University of Buffalo
B. H. Gere, Hamilton College
Michael Goldberg, Bureau of Ordnance, Navy Department
J. R. Gorman, U. S. Naval Academy
Lillian Gough, University of Buffalo
G. F. Guilford, Jr., Rensselaer Polytechnic Institute
Lucille F. Hetzelt, Syracuse University
H. K. Holt, Union College
R. E. Huston, Rensselaer Polytechnic Institute
Roberta F. Johnson, Wilson College
A. W. Jones, Rensselaer Polytechnic Institute
H. K. Justice, University of Cincinnati
F. E. Justis, Geneva College
Aida Kalish, Polytechnic Institute of Brooklyn
W. C. Krathwohl, Illinois Institute of Technology
John Kronsbein, Evansville College
R. E. Langer, University of Wisconsin
Caroline A. Lester, N. Y. State College for Teachers at Albany
M. E. Levenson, Cooper Union
J. V. Limpert, St. Lawrence University
June M. McArtney, University of Buffalo
V. O. McBrien, College of the Holy Cross
Dis Maly, Rensselaer Polytechnic Institute
L. L. Merrill, Clarkson College
A. B. Mewborn, U. S. Naval Postgraduate School
F. H. Miller, Cooper Union
Norman Miller, Queen's University
W. L. Miser, Vanderbilt University
R. K. Morley, Worcester Polytechnic Institute
D. S. Morse, Union College
Abba V. Newton, Vassar College

RUTH B. NOLLER, University of Buffalo
F. S. NOWLAN, University of Illinois
MARY H. PAYNE, Michigan State College
K. S. PURDIE, Virginia Military Institute
J. F. RANDOLPH, University of Rochester
W. R. RANSOM, Tufts College
G. B. ROBISON, Sampson College
I. H. ROSE, University of Massachusetts
M. F. ROSSKOPF, Syracuse University
S. G. ROTH, New York University
W. E. ROTH, University of Tulsa
EDITH R. SCHNECKENBURGER, University of Buffalo
WILLIAM A. SMITH, St. Lawrence University
I. S. SOKOLNIKOFF, University of California at Los Angeles
ELLEN C. STOKES, N. Y. State College for Teachers at Albany
R. R. STOLL, Lehigh University
J. S. TAYLOR, University of Pittsburgh
C. J. THORNE, University of Utah
BRYANT TUCKERMAN, Cornell University
S. I. VROOMAN, Rensselaer Polytechnic Institute
R. J. WALKER, Cornell University
W. G. WARNOCK, Rensselaer Polytechnic Institute
BERNICE L. WARR, General Electric Company
MARGARET C. WEEBER, Teachers College of Connecticut
P. M. WHITMAN, Johns Hopkins University
R. H. WILSON, JR., Temple University
J. H. ZANT, Oklahoma A & M College
H. M. ZERBE, Hazleton Center, Pennsylvania State College

The Mathematical Association of America held its first session on Monday afternoon in Sage Lecture Hall with Professor F. H. Miller, Chairman of the Mathematics Division, A.S.E.E., presiding. Professor L. R. Ford presided at the banquet on Monday evening in the Snack Bar. The second session was also held in Sage Lecture Hall on Tuesday afternoon with President R. E. Langer presiding. The Program Committee consisted of J. S. Taylor, Chairman, Michael Goldberg, and R. J. Walker.

FIRST SESSION

"Faulty Teaching of Mathematics," by Dr. L. B. Tuckerman, National Bureau of Standards.

"Mathematical Thinking," by Professor H. B. Phillips, Massachusetts Institute of Technology.

BANQUET

"Mathematical Obligations of the Engineer," by Professor Arnold Dresden, Swarthmore College.

SECOND SESSION

Retiring presidential address: "The Geometry of the Sliding Plane," by Professor L. R. Ford, Illinois Institute of Technology.

"The Use of Slides in the Teaching of College Mathematics," by Professor J. W. Cell, North Carolina State College.

"Real and Complex Vector Analysis," by Professor W. E. Restemeyer, University of Cincinnati.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Monday evening in the Trustees Room, Pittsburgh Building. Eight members of the Board were present.

Reports of various committees were received and acted upon. Among other items of business, the Board voted that the Index of volumes 1–55 of the MONTHLY be published as a supplement to the MONTHLY and be distributed without charge to all members and subscribers. It was further noted that the expenses of editing and printing the Index be charged to the Jacob Houck Memorial Fund.

SOCIAL EVENTS

Local arrangements for the meeting were cared for by the members of the Department of Mathematics at Rensselaer Polytechnic Institute under the capable chairmanship of Professor E. B. Allen. At a brief business meeting of the Association held on Tuesday afternoon, the assembled mathematicians voted an expression of their appreciation to the authorities of the Institute, especially to Professor Allen and the other members of the local committee on arrangements.

A reception for all in attendance at the meetings was held in the Clubhouse on Monday evening. An exhibit of paintings from the Fine Arts Department of I.B.M. was on display at this time.

On Tuesday afternoon at the conclusion of the session of the Association, the visiting mathematicians were entertained at a tea in the Ball Room of the Clubhouse.

Sessions of the A.S.E.E. began on Monday and continued through Friday. In addition to the sessions mentioned above, the Mathematics Division of the A.S.E.E. met jointly with the Mechanics Division on Wednesday afternoon.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred one persons have been elected to membership by the Board of Governors on applications duly certified:

JANET E. ABBEY, B.A. (William Smith) Instructor, University of Buffalo, N. Y.

A. D. ANDERSON, M.A. (Oregon) Student, West Branch, Iowa.

H. A. ANTOSIEWICZ, Ph.D. (Vienna) Asst. Professor, Montana State College, Bozeman, Mont.

W. H. BADGLEY, JR., M.A. (Columbia) Asst. Professor, Florence S.T.C., Ala.

R. W. BALL, Ph.D. (Illinois) Instructor, University of Washington, Seattle, Wash.

Jo. M. BALLARD, B.S. (Howard) Instructor, Howard College, Birmingham, Ala.

E. F. BARTLEY, B.A. (Scranton) Asst. Professor, University of Scranton, Pa.

O. K. BATES, Sc.D. (M.I.T.) Professor, St. Lawrence University, Canton, N. Y.

W. R. BECK, M.A. (N.Y.U.) Instructor, Purdue University, Fort Wayne, Ind.

F. S. BECKMAN, A.M. (Columbia) Instructor, Pratt Institute, Brooklyn. N. Y.

A. I. BENSON, M.S. (Wisconsin) Staff Member, Los Alamos Scientific Laboratory, N. M.

L. E. BERG, M.A. (Syracuse) Asst. Professor, North Georgia College, Dahlonega, Ga.

IDA M. BERNHARD, M.A. (Texas) Supervisor of Mathematics, Southwest Texas S.T.C., San Marcos, Texas.

NANETTE R. BLACKISTON, M.A. (Columbia) Supervisor, Department of Education, Baltimore, Md.

W. R. BLANK, A.B. (Union) Instructor, Union College, Lincoln, Nebr.

J. L. BRENNER, Ph.D. (Harvard) Asst. Professor, University of California, Santa Barbara College, Calif.

N. A. BRIGHAM, Ph.D. (Pennsylvania) Asst. Professor, University of Maryland, College Park, Md.

B. F. BRYANT, M.A. (Peabody) Instructor, Vanderbilt University, Nashville, Tenn.

I. A. CARL, M.A. (Columbia) Asst. Professor, New York University, N. Y.

C. L. CARROLL, JR., Ph.D. (North Carolina) Asso. Professor, North Carolina State College, Raleigh, N.C.

H. W. CHARLESWORTH, M.A. (Colorado State College of Education) Chairman of Department, Denver Public Schools, Colo.

W. G. CLARK, Ph.D. (Kentucky) Asst. Professor, Mount Union College, Alliance, Ohio

W. H. CLEVELAND, M.S. (Alabama) Teacher, Meridian Junior College, Miss.

RIA J. CLINKSCALES, M.A. (Alabama) Instructor, University of Alabama, Montgomery, Ala.

K. L. COOKE, M.S. (Stanford) Student, Stanford University, Calif.

PATRICIA M. COWAN, B.A. (Reed) Student, Reed College, Portland, Ore.

MARY L. CUMMINGS, M.A. (Illinois) Instructor, University of Missouri, Columbia, Mo.

MORRIS DANSKY, M.A. (Michigan) Instructor, Creighton University, Omaha, Nebr.

D. B. DEKKER, Ph.D. (California) Instructor, University of Washington, Seattle, Wash.

R. D. DEPEW, M.A. (Peabody) Asst. Professor, Florence S.T.C., Ala.

THOMAS DUMONT, Lt. Comdr., Marine Inspector, U. S. Coast Guard, Albany, N. Y.

MARSHALL ELDER, M.A. (Southern California) Instructor, Los Angeles City College, Calif.

TERRELL ELLIS, M.A. (Texas Christian) Asst. Professor, North Texas State College, Denton, Texas

MR. GERALDINE GALLOWAY, M.A. (Illinois) Head of Department, Flat River Junior College, Mo.

H. E. GERTZ, B.S. (Illinois Institute of Technology) Grad. Student, Illinois Institute of Technology, Chicago, Ill.

R. T. GREGORY, M.S. (Iowa) Instructor, Florida State University, Tallahassee, Fla.

D. K. HARTMAN, M.S. (Minnesota) Instructor, University of Minnesota, Minneapolis, Minn.

R. G. HILL, Student, University of Buffalo, N. Y.

H. E. HOFFMAN, Student, University of Buffalo, N. Y.

R. E. HOGAN, B.S. (Southwest Missouri State College) Asst. Instructor, University of Missouri, Columbia Mo.

O. H. HOKE, B.S. (Franklin and Marshall) Grad. Student, University of North Carolina, Chapel Hill, N. C.

JULIUS HONIG, B.S. (Michigan) Student, University of Michigan, Ann Arbor, Mich.

A. J. KILLEBREW, M.S. (Auburn) Asso. Professor, S.T.C., Livingston, Ala.

C. E. KIRKWOOD, JR., M.S. (Georgia) Asso. Professor, Clemson A & M College, S. C.

HELENE G. KUSICK, A.B. (California) Teacher, Placer Union High School, Auburn, Calif.

L. M. LARSEN, M.A. (Nebraska) Professor, Kearney State College, Nebr.

EUGENE LEIMANIS, Dr. Rer. Nat. (Hamburg) Asst. Professor, University of British Columbia, Vancouver, B. C.

C. E. LEMKE, B.A. (Buffalo) Student, University of Buffalo, N. Y.

P. J. LEONARD, B.S. (Boston College) Grad. Student, Boston College, Mass.

E. R. LORCH, Ph.D. (Columbia) Professor, Barnard College, Columbia University, New York, N. Y.

R. L. MARCEAU, Licence en Sciences (Laval University) Instructor, University of Kansas, Lawrence, Kansas.

MILDRED M. MARTENS, M.A. (Michigan) Instructor, Woodruff Senior High School, Peoria, Ill.

D. E. MCCLEAN, B.A. (George Pepperdine) Grad. Student, University of Southern California, Los Angeles, Calif.

R. K. MCCONNELL, JR., B.S. (Pittsburgh) Instructor, New York University, N. Y.

ELOISE MCCORD, M.A. (Oregon) Instructor, University of Wichita, Kansas

D. L. MCINTOSH, M.A. (Denver) Teacher, South Denver High School, Colo.

J. C. C. MCKINSEY, Ph.D. (California) Rand Corporation, Santa Monica, Calif.

RUDOLF MEYER, B.A. (Buffalo) Student, University of Buffalo, N. Y.

L. C. MILLS, M.A. (Harvard) Teacher, Wheat Ridge High School, Colo.

MOTHER M. GERARD MOONEY, M.A. (Creighton) Teacher, Ursuline College, New Orleans, La.

H. W. NACE, M.A. (Cornell) Professor, Lawrence Institute of Technology, Detroit, Mich.

A. R. NOLSTAD, D.Ed. (Pittsburgh) Asst. Professor, North Carolina State College, Raleigh, N. C.

HELEN OLNEY, M.Sc. (Oregon) Asst. Professor, Hiram College, Ohio

M. H. PROTTER, Ph.D. (Brown) Asst. Professor, Syracuse University, N. Y.

F. M. PULLIAM, Ph.D. (Illinois) Asst. Professor, University of Kentucky, Lexington, Ky.

A. H. QUIRMBACH, M.S. (Virginia Polytechnic Institute) Asst. Professor, University of Alabama, University, Ala.

L. B. RALL, Student, College of Puget Sound, Tacoma, Wash.

R. F. REEVES, B.S., Instructor, Elec. Engg., Iowa State College, Ames, Iowa.

L. A. RINGENBERG, Ph.D. (Ohio State) Professor, Eastern Illinois State College, Charleston, Ill.

E. K. RITTER, Ph.D. (Virginia) Research Engineer, University of Michigan, Ypsilanti, Mich.

SIBYL M. ROCK, B.A. (California) Technical Consultant, Consolidated Engineering Corporation, Pasadena, Calif.

A. I. ROSENFELD, 475 W. 186th St., New York, N. Y.

E. D. ROUNDS, B.S. (Scranton) Instructor, University of Scranton, Pa.

J. E. ROWE, B.A. (Tennessee) Research Engr., Carbide and Carbon Chemical Corporation, Oak Ridge, Tenn.

W. C. ROYSTER, M.A. (Kentucky) Instructor, University of Kentucky, Lexington, Ky.

C. M. SANDWICK, B.A. (Lehigh) Teacher, Easton High School, Pa.

P. J. SCHILLO, Student, University of Buffalo, N. Y.

L. F. SCHOLL, M.A. (Buffalo) Supervisor of Math., Board of Education, Buffalo, N. Y.

D. M. SEWARD, Ph.D. (Duke) Professor, Ouachita College, Arkadelphia, Ark.

SISTER M. AGNETA, B.A. (St. Teresa) Teacher, St. Clare Academy, Sylvania, Ohio

SISTER ROSE G. CALLOWAY, Ph.D. (Catholic University) Professor, Mt. St. Mary's College, Los Angeles, Calif.

WILLIAM A. SMITH, M.A. (Syracuse) Asst. Professor, St. Lawrence University, Canton, N. Y.

R. J. SMURTHWAITE, Student, University of Buffalo, N. Y.

E. J. SPECHT, Ph.D. (Minnesota) Professor, Emmanuel Missionary College, Berrien Springs, Mich.

H. S. STANLEY, A.M. (Harvard) Asso. Professor, University of Georgia, Athens, Ga.

S. N. STONE, A.B. (Brooklyn) Grad. Student, New York University, N. Y.

R. L. SWAIN, Ph.D. (Texas) Instructor, Ohio State University, Columbus, Ohio

G. J. TRAMMELL, JR., B.S. (Tulane) Student, Tulane University, New Orleans, La.

T. E. TUTHILL, M.A. (Oberlin) Teacher, Nichols School, Buffalo, N. Y.

J. P. VAN ALSTYNE, B.S. (Hamilton) Instructor, Hamilton College, Clinton, N. Y.

E. L. VANDERBURGH, M.A. (Wyoming) Instructor, Pueblo Junior College, Colo.

E. A. VOORHEES, JR., A.B. (Maryville) Fellow Vanderbilt University, Nashville, Tenn.

W. E. VORONOVICH, Graduate (Polytechnic Institute of Riga) Lewis Machine Co., Cleveland, Ohio

SYLVIA VOPNI, M.A. (University of Washington) Instructor, Edison Technical School, Seattle, Wash.

J. F. WAGNER, M.S. (Michigan) Asst. Professor, University of Colorado, Boulder, Colo.

M. C. WALKER, B.S. (Illinois) Instructor, General Motors Institute, Flint, Mich.

MARGUERITE F. WELLS, M.A. (Alabama) Statistician, Osborn, Ohio

J. R. WESSON, B.S. (Birmingham-Southern) Teaching Fellow, Vanderbilt University, Nashville, Tenn.

MABEL WILLIAMS, M.A. (Texas) Teacher, Tyler Junior College, Texas

W. L. G. WILLIAMS, Ph.D. (University of Chicago) Professor, McGill University, Montreal, P. Q.

R. B. WINEGEART, Student, Northwestern State College, Natchitoches, La.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1949 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	R. G. Sanger, Kansas State College
Missouri	G. M. Ewing, University of Missouri
Ohio	F. B. Wiley, Denison College
Pacific Northwest	M. S. Knebelman, State College of Washington
Southeastern	Tomlinson Fort, University of Georgia
Southwestern	Earl Walden, New Mexico College of A. and M. A.
Upper New York State	E. B. Allen, Rensselaer Polytechnic Institute
New England Region	R. E. Gilman, Brown University

Since the system of Regional Governors has been replaced by a system of Sectional Governors, no elections have been held (except in the New England Region) to replace the Regional Governors whose terms expired on July 1, 1949.

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The eleventh annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco on Saturday, January 29, 1949. Professor G. C. Evans, Chairman of the Section, presided at the morning session, and Mr. K. J. Waider presided at the afternoon session.

The attendance was eighty-one including the following thirty-five members of the Association: H. M. Bacon, G. A. Baker, T. J. Bass, Alice Bell, M. T. Bird, A. C. Burdette, D. G. Chapman, M. A. Dernham, Roy Dubisch, Hazel Eggett, G. C. Evans, E. A. Fay, S. A. Francis, C. M. Fulton, W. R. Hanson. V. F. Ivanoff, D. H. Lehmer, Sophia McDonald, E. D. Miller, F. R. Morris, W. H. Myers, C. D. Olds, George Polya, Edris Rahn, R. M. Robinson, E. B. Roessler, Kathryn Rolfe, Sister Madeleine Rose, Irving Sussman, Gabor Szego, Edwin Tabor, K. J. Waider, L. A. Walker, Anna Pell Wheeler, A. R. Williams.

At the business meeting the following officers were elected for the coming year: Chairman, H. M. Bacon, Stanford University; Vice-Chairman, S. A. Francis, San Mateo Junior College; Secretary-Treasurer, E. B. Roessler, University of California at Davis; Representative on the *California Journal of Secondary Education*, Ruth G. Sumner, Oakland High School.

By invitation of the Section, Professor Edward W. Strong, Department of Philosophy, and Associate Dean of the College of Letters and Science, University of California, gave an hour's address during the morning session.

1. *The irrationality of a class of numbers*, by Dr. H. L. Alder, University of California at Davis, introduced by the Secretary.

A method is given by which to prove the irrationality of a class of numbers including e^m, where m is rational and not equal to zero, in such a way as to be understandable to students with only a little knowledge of differential and integral calculus and no knowledge of series expansions.

2. *With, or without, motivation?* by Professor George Polya, Stanford University.

This paper will be published in this MONTHLY.

3. *Newton's "mathematical way,"* by Professor E. W. Strong, University of California, introduced by the Chairman.

Newton's procedure in "mathematically determining all kinds of phenomena" presents two problems: (1) how does he explain "principles" as inductive generalizations from which a theoretical physicist proceeds mathematically to "estimate effects"? (2) what is his position concerning terms that are strictly mathematical?

An examination of his work in optics reveals the central role assigned to measurement in the formulation of mechanical (mathematical-physical) principles. The rules derived from measures in experimental inquiry and incorporated in the statement of laws are requisite for demonstration of phenomena. Newton insists that the certainty of demonstration in science cannot exceed the certainty of its physical principles. An exception to his argument appears to be his assertion of absolute, true, and mathematical space, time, and motion in the *Principia*; yet Newton himself is cognizant that he is here postulating a mathematical structure lacking in empirical support.

Although he devised his method of fluxions as a tool to assist in the solution of physical problems, Newton treats mathematical analysis *per se* as a logic of reasoning advancing concepts requiring no appeal to geometry or to nature for their legitimacy. The reasoning in "a method of determining quantities from the velocities of the motions of the increments with which they are generated" assigns meanings to the terms employed that are strictly mathematical. Subsequently a number of British mathematicians attempted to argue the superiority of Newton's method to Leibniz's differential calculus. Their defense tried not only to deny that Newton had ever employed infinitesimals but also rejected their use on the grounds that they "have no being either in geometry or in nature." Berkeley turned this argument against the defenders of Newton and was instrumental in prodding Robins, Simpson, and Maclaurin to investigate the foundations of the calculus in contributing to the theory of limits.

4. *Probability and nature*, by Professor Michel Loeve, University of California, introduced by the Chairman.

A few examples, chosen for their historical importance, illustrated the coming insight of probability notions and their gradual penetration into the rational models for natural phenomena. At first their domain seemed to be that of mass-phenomena such as the kinetic theory of gases, statistical mechanics, and genetics. At present, after a revolution in the understanding of nature, due to quantum mechanics, even individual microscopic phenomena are best explained in probability terms.

5. *Cayley arrays and multiplication matrices of an algebra*, by Roy Dubisch, Fresno State College.

Considering an algebra A over a field F as a linear space of order n over F consisting of 1 by n matrices $a=(\alpha_1, \cdots, \alpha_n)$, α_i in F, we define multiplication in A by $a \cdot x=(\alpha_1, \cdots, \alpha_n)(\xi_1, \cdots, \xi_n)=(\mu_1, \cdots, \mu_n)$, where $\mu_k=\sum_{i,j=1}^{n}\alpha_i\gamma_{ijk}\xi_j$ for $k=1, \cdots, n$. The right multiplication matrix of A is defined as $\Gamma_x=(\sum_{k=1}^{n}\gamma_{ijk}\xi_k)$ while the matrix formed from the Cayley array is defined to be $\Omega_x=(\sum_{k=1}^{n}\gamma_{ijk}\xi_k)$. We ask what algebras have $\Gamma_x=\Omega_x$ for all possible bases, and find that A has such a property if and only if A has a basis $u_1, u_2, \cdots, u_n$ with $u_su_r=\delta_su_r$ (δ_s in F, $s, r=1, \cdots, n$).

6. *A note on linear equations*, by Professor R. M. Robinson, University of California.

This note has since appeared in this MONTHLY, vol. 56, April, 1949, p. 251.

7. *Geometrical optics and the calculus of variations*, by Dr. Robert Weinstock, Stanford University, introduced by the Vice-Chairman.

It is shown how certain elementary minimum problems may be solved through the use of geometrical optics in conjunction with Fermat's principle of least time.

Further, the simple laws of geometrical optics are applied to a medium in which the index of refraction is a continuous function of one cartesian coordinate, through "slabification" and suitable passage to the limit. Since the optics problem, through Fermat's principle, is equivalent to a certain class of problems in the Calculus of Variations, the latter may be solved by optical means alone. For example, the classical brachistochrone problem is solved upon suitable choice of index of refraction as a function of one coordinate (John Bernoulli's original method of solution).

Finally, the differential equation in plane polar coordinates for the path of light in a medium whose index is a function of the radial coordinate alone is produced as a solution to another class of elementary problems in the Calculus of Variations.

E. B. ROESSLER, *Secretary*

FEBRUARY MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Friday, February 18, 1949. Professor J. E. LaFon, Chairman of the Section, presided.

Sixty-two persons attended the meeting, including the following thirty members of the Association: E. F. Allen, Arthur Bernhart, J. C. Brixey, A. H. Diamond, R. C. Dragoo, A. A. Grau, L. D. Gregory, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. E. Heimann, L. R. Holland, W. N. Huff, H. V. Huneke, J. T. Krattiger, M. L. Lawson, Eunice Lewis, H. W. Linscheid, R. D. McDole, Dora McFarland, R. D. McKnelly, G. E. Meador, R. R. Murphy, C. J. Pipes, A. A. Ritchie, H. W. Smith, O. S. Spears, C. E. Springer, R. W. Veatch, J. H. Zart.

At the business session the following officers were elected: Chairman, R. W. Veatch, University of Tulsa; Vice-Chairman, E. E. Heimann, East Central State College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following seven papers:

1. *Some aspects of teaching calculus*, by Professor Caspar Goffman, University of Oklahoma, introduced by Professor J. C. Brixey.

The speaker discussed a method of presenting the fundamental theorem of the calculus, and its applications, without using the theorem of the mean.

2. *Semi-continuous transformations*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

The definitions and some of the properties of upper and lower semi-continuous transformations were discussed. Comparisons of semi-continuous with continuous transformations were made, and examples of the various types of transformations were given.

3. *Set properties*, by Mr. C. J. Pipes, University of Oklahoma.

For sets of real numbers, a set property is called a τ property if every subset of a τ set is a τ set, the union of a denumerable number of τ sets is a τ set, the continuum is not a τ set, and there is at least one non-empty τ set. A set property is called a π property if it is a τ property and if every set consisting of a single point is a π set, and it is a ρ property if it is a π property and if there are C or less ρ sets such that every ρ set is a subset of one of them, where C is the cardinal of the continuum. With these set properties various theorems on point sets can be generalized; e.g., "there is a non-denumerable set whose intersection with every ρ set is denumerable."

4. *Some metric properties of union curves on a surface*, by Mr. R. B. Deal, Jr., University of Oklahoma, introduced by Professor C. E. Springer.

The contour integral of union curvature was considered, and a generalization of the Gauss-Bonnet theorem was given. The torsion of any curve was expressed in terms of its union torsion, and for a one-parameter family of curves the union torsion was given, as well as necessary and sufficient conditions for the curves to be union curves relative to a specified rectilinear congruence. Finally, necessary and sufficient conditions were given for two surfaces to be in union correspondence.

5. *What colleges can do to improve mathematics in the public schools*, by Professor E. E. Heimann, East Central State College.

It was pointed out that as long as colleges continue to offer remedial mathematics courses without holding the high schools to account for the poor mathematical training of their graduates, the high schools may be expected to continue to shirk their responsibility. In addition it was stated that elementary teachers should not be allowed to teach arithmetic without having had some college mathematics.

6. *Binary and ternary relations*, by Professor A. A. Grau, University of Oklahoma.

In this paper, the author showed that the role played by the ternary relation $(a, b, c) = (a \cap b) \cup (a \cap c) \cup (b \cap c) = a$, denoted herein by $R(a; b, c)$, in a ternary Boolean algebra or lattice, is analogous to that of the partial order relation in a binary lattice. The relation $R(a; b, c)$ has the following properties: (1) it is symmetric in b and c; (2) it is reflexive in any pair; (3) for fixed b or c, it is a binary partial order relation; (4) $R(a; a', b)$ never holds if $a \neq b$; (5) $R(b; a, a')$ always holds; (6) if $R(x; a, b)$, $R(x; b, c)$, $R(x; a, c)$ hold simultaneously in a Boolean algebra, $x = (a; b, c)$. If properties (1)–(3) are used as postulates, (4) and (5) as the definition of complements (whenever they exist), and (6) as the definition of the ternary operation, many of the properties of the ternary operation in a Boolean algebra or lattice may be derived from the properties of $R(a; b, c)$.

7. *A property of a class of parabolic curves*, by Professor A. H. Diamond, Oklahoma A. and M. College.

The radius of curvature R of the class of parabolic curves $y = \pm x^n$, $x > 0$ and $0 < n < 1$, was examined at the origin. It was pointed out that R considered as a function of n at the origin, has the discontinuity $0 < n < 0.5$, $R = \infty$; $n = 0.5$, $R = 1$; $0.5 < n < 1$, $R = 0$.

J. C. BRIXEY, *Secretary*

MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Alabama, University, Alabama, on Friday and Saturday, March 18–19, 1949. Major L. A. Dye, Chair-

man of the Section, presided at the Friday afternoon and Saturday morning meetings, except for the meetings of the subsections, which were presided over by Professors F. W. Kokomoor, J. M. Thomas, and M. G. Boyce. Professor F. A. Lewis, Vice-Chairman, presided on Friday evening at the barbecue held at Moundville State Park.

There were about two hundred present, including the following one hundred and six members of the Association: R. H. Ackerson, W. H. Badgley, Jr., Edna M. Ballard, D. F. Barrow, W. S. Beckwith, L. E. Berg, R. G. Blake, Floyd Bowling, M. G. Boyce, Mamie I. Braswell, J. P. Brewster, S. K. Bright, J. W. Brown, C. W. Bruce, N. R. Bryan, B. F. Bryant, Berdie E. J. Buffkin, L. P. Burton, John Capesius, Ella F. Casey, Elizabeth C. Cathey, B. G. Clark, Ria J. Clinkscales, A. C. Cohen, Jr., R. L. Coker, G. M. Conwell, J. A. Cooley, R. W. Cowan, J. C. Currie, R. D. Depew, R. D. Doner, Nelle C. Douglas, L. A. Dye, E. D. Eaves, J. D. Edwards, R. B. Folsom, Tomlinson Fort, S. T. Gormsen, W. W. Graham, E. H. Hadlock, B. F. Hadnot, E. A. Hedberg, G. W. Hess, A. T. Hind, Jr., G. B. Huff, P. M. Hummel, R. O. Hutchinson, J. A. Hyden, Rosa L. Jackson, Ayrlene M. Jones, A. J. Killebrew, F. W. Kokomoor, W. I. Layton, R. J. Levit, F. A. Lewis, J. F. Locke, G. H. Lundberg, J. D. Mancill, J. E. Martin, W. A. Martin, W. G. McGavock, S. W. McInnis, W. L. Miser, W. A. Moore, R. H. Moorman, T. F. Mulcrone, W. V. Neisius, J. D. Novak, H. A. Palmer, Eugene Park, W. V. Parker, Lillian G. Perkins, I. E. Perlin, C. G. Phipps, R. B. Plymale, A. H. Quirmbach, Alice B. Rabon, Adrienne S. Rayl, B. P. Reinsch, G. E. Reves, J. O. Reynolds, R. G. D. Richardson, H.A. Robinson, L. V. Robinson, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, T. M. Simpson, Augustus Sisk, A. R. Sloan, C. B. Smith, C. D. Smith, E. L. Stanley, H. S. Stanley, R. B. Stiles, J. R. Sullivan, J. M. Thomas, E. A. Voorhees, F. A. Wallace, J. A. Ward, Betty R. Weber, W. W. Weber, W. L. Williams, R. L. Wilson, G. N. Wollan, F. L. Wren.

At the business session the following officers were elected for the coming year: Chairman, F. A. Lewis, University of Alabama; Vice-Chairman, C. G. Phipps, University of Florida; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its March 1950 meeting at the University of Florida.

The program consisted of the following papers:

1. *Applications of statistics in the transportation industry*, by W. V. Neisius, Consultant, Transportation Statistics, Georgia Power Company.

A description of the factors used in a multiple curvilinear correlation analysis, as developed by C. A. Stephenson, and applied to city transit rides, was given. Probable annual rides per capita were described very accurately in terms of average fare, miles of service offered, automobile registration, and bank debits.

2. *Nation-wide practices in the certification of teachers of mathematics*, by Professor W. I. Layton, Alabama Polytechnic Institute.

The present requirements for certification of teachers were reviewed, and recommendations

were offered for the possible improvement of the chaotic conditions now prevailing. It was suggested that six semester hours in mathematics be required as a minimum for certification of an elementary teacher, and eighteen for a secondary school teacher.

3. *The approximate evaluation of roots*, by Professor L. A. Dye, The Citadel,

In this expository paper, four standard procedures for approximating irrational roots of an equation were discussed. Three simple rules for gaining greater accuracy when using Newton's or Horner's methods were developed.

4. *The role of fields in college mathematics courses*, by Professor G. B. Huff. University of Georgia.

Professor Huff remarked it is a common complaint that college mathematics courses rarely offer the student much opportunity to formulate arguments, and suggested a possible remedy. After reviewing briefly some fundamental notions in the theory of abstract fields, he showed how these ideas recur in the usual material and thus present chances to make arguments.

5. *Mathematics and economics: a preliminary report*, by Professor C. G. Phipps, University of Florida.

In recent years, mathematics has been increasingly applied to problems in theoretical economics entirely apart from economic statistics. Unfortunately, some writers have desired to add prestige to their conclusions by giving mathematical proofs, but they have been reluctant to be bound by mathematical rigor. In order to establish this rigor, a graduate project is being set up at the University of Florida with a twofold purpose: (a) to make a complete and accurate analysis of certain topics in economics; and (b) with this analysis as a criterion, to review critically books and articles dealing with these topics.

6. *Plane areas by complex integration*, by Professor J. D. Mancill, University of Alabama.

This paper is to appear in this MONTHLY.

7. *Logarithms and exponentials in calculus*, by Professor Tomlinson Fort, University of Georgia.

The differentiation of the logarithm and exponential is a traditionally difficult matter for writers of textbooks on calculus. The fundamental difficulty is that these functions have never been defined in the mathematical career of the student for irrational values of the argument, and consequently any limiting process involving them is necessarily incomplete. However, if the theory of integration is properly presented before the introduction of logarithms and exponentials into calculus, the whole procedure is greatly simplified. Thus $\int_1^s (1/x)dx$ is a well defined function of s for irrational as well as rational values of s.

8. *Bernoulli's asterisks*, by P. A. Piza, San Juan, Puerto Rico.

In this paper, which Mr. Piza had printed for the occasion of the meeting, he makes certain deductions concerning the asterisks which appear in Bernoulli's formulae for the sums S_n of the first nth powers, as given on Page 97 of *Ars Conjectandi*. Mr. Piza demonstrates two methods whereby these formulae may be simplified and developed without employing Bernoulli numbers. He also develops a number of syzygies connecting certain S_i's.

9. *Conclusions from a survey on mathematics curricula in liberal arts colleges*, by Professor N. R. Bryan, University of Georgia.

This survey covers curricula conditions of the last ten years. One trend is to include more

differential equations as subject matter in undergraduate mathematics. Calculus is begun earlier by putting the differentiation and integration of algebraic functions in the freshman year. A much briefer course in analytic geometry is made possible by postponing such topics as polar coordinates and solid analytics to the calculus, where these topics can be freshly taught when needed.

10. *A novel approach to the addition formulae of trigonometry*, by Professor E. A. Hedberg, University of South Carolina.

This approach is designed to motivate and rationalize the development of the addition formulae of trigonometry by making use of the laws of sines and cosines. A triangle is constructed with given angles $A+B$ and $90°-A$, and the proof follows immediately.

11. *Mathematics in the New England private schools*, by Professor G. M. Conwell, University of Georgia.

Professor Conwell outlined the objectives of these schools, and told how they attempted to secure for the student an understanding and love of mathematics. These schools try to test ability rather than memory. Emphasis is placed on understanding. All students take algebra and geometry, the better ones solid geometry, trigonometry and advanced algebra. A continuous effort is made to use algebra in geometry and geometry in algebra. The best boys take a year of calculus.

12. *The need for cooperative effort in mathematics*, by Professor F. L. Wren, Peabody College.

In recent years we have witnessed a very definite increase in the demand for an effective program in the teaching of mathematics. This fact confronts in particular the Mathematical Association of America and the National Council of Teachers of Mathematics, with the distinct challenge to determine what this program should be. Is the program in our elementary and secondary schools the most effective that we can determine, or is it somewhat an accident of tradition? What is the most significant program we can shape for the training of teachers? Such questions can be answered authoritatively only through the cooperative efforts of the interested organizations.

13. *Teaching through discovery*, by Professor I. E. Perlin, Georgia Institute of Technology.

In this paper the importance of examining definitions and basic concepts critically was stressed. It was demonstrated that the best way to introduce a basic concept is, if possible, to let the student discover the method himself, with the instructor only acting as a guide along the proper paths. It was felt that this procedure might develop a way of thinking of great benefit to the scientific man.

14. *Grading mathematics papers*, by Professor D. F. Barrow, University of Georgia.

The grading of quiz papers is too often regarded as a tedious chore. Professor Barrow suggested some devices for making the task more interesting and more valuable to the students, and which might to some extent prevent instructors from falling into a rut.

15. *A new roll book system*, by W. V. Neisius, Georgia Institute of Technology.

A simplified visual index roll book was described which enables an instructor to determine graphically the student's average. The new roll book system is of great benefit in large classes where frequent quizzes are given.

16. *Sample design for the study of advertising*, by Professor C. D. Smith, University of Alabama.

Professor Smith discussed factors of significance in advertising cost. Data were exhibited for a sample pattern applied to a given area where one spends a cumulative total X to obtain an increase in net profit M on sales.

17. *A distinguished mathematician's contribution to genetics*, by Professor R. J. Levit, University of Georgia.

Shortly after the rediscovery of Mendel's law in 1900 a misconception arose regarding the distribution of hereditary traits among a population to be expected on the basis of these laws. This speaker discussed a solution of the problem by the application of the elementary theory of probability as given by the late G. H. Hardy in a letter to the editor of *Science* in 1908, out of which has grown the modern science of statistical genetics.

18. *On teaching the law of means*, by Professor B. P. Reinsch, Florida Southern College.

Professor Reinsch shared his experiences in motivating interest and understanding on the part of the student in connection with the law of the mean for derivatives. He demonstrated a method for constructing tangents to parabolas.

19. *Algebraic methods of extending the multiplication table*, by Professor L. V. Robinson, University of South Carolina.

It was shown how the simple principles of algebra may be applied to oral multiplication and division, and how interest in algebra may thereby be enhanced.

20. *The number of solutions of Diophantine systems*, by Professor J. M. Thomas, Duke University.

For real x the four symbols $[x]$, $[x]'$, $\{x\}$, $\{x\}'$ are defined respectively as the greatest integer not greater than x, the least integer not less than x, the least integer nearest x, and the greatest integer nearest x. Relations among these symbols are given, and the number of solutions of certain systems composed of Diophantine equations and inequalities is expressed in terms of them. In particular, the paper treats the system composed of a linear Diophantine equation and inequalities restricting the unknowns to segments or intervals.

21. *Elastic equilibrium for the interior of a wood disk*, by C. B. Smith, University of Florida.

The wood disk was assumed to be in a state of plane stress under the action of external forces which are applied to the boundary, and which act in the plane of the disk. A method was developed for solving the problem for a general distribution of external forces. Then the problem was carried out in detail for a special type of loading, and the results reduced to the case where the disk was of isotropic material.

22. *A refinement for linear interpolation applied to approximations of the roots of equations*, by Professor C. L. Seebeck, Jr., University of Alabama.

The refinement is additive and equal to

$$(b - x)(x - a)[f'(a) - f'(b)]/2(b - a),$$

where $b-a$ is the interpolation interval, and x is between a and b. When applied to the inverse of $y=f(x)$, a formula results for improving the approximate root of $f(x)=0$ obtained by linear interpolation. Rapid convergence toward roots was exhibited in several illustrations.

23. *On properties of particular solutions of a generalized Bessel's equation*, by Professor R. W. Cowan, University of Florida.

The equation is taken in a form so that it contains two parameters n and α such that when α is put equal to zero, the equation reduces to Bessel's equation. The general solution of the equation is obtained by the method of Frobenius. A number of properties of a particular solution $G_{n,\alpha}(x)$ are obtained including a recussion formula, orthogonality relations, and the evaluation of the integrated square. The latter two properties enable one to expand an arbitrary function in a series of $G_{n,\alpha}$ functions, each of which is multiplied by a certain exponential factor.

24. *On the numerical solution of a transcendental equation in statistics*, by Professor A. C. Cohen, Jr., University of Georgia.

In order to fit a truncated normal distribution to a given set of data, it is necessary to solve a somewhat complicated type of transcendental equation. In this paper, the fundamental equations involved in solutions previously given by Karl Pearson, Alice Lee, and R. A. Fisher are expressed in terms of more elementary transcendental functions and written in a form which facilitates their rapid computation. Using only an ordinary set of tables of areas and ordinates of the normal frequency curve, the solution can be completed by the simple method of successive graphs or interpolation.

25. *A note on the general polynomial theorem*, by Professor W. S. Beckwith, University of Georgia.

The purpose of this paper was to set up a general method for the expansion of a sum of algebraic variables raised to a power. A general theorem was developed by repeatedly employing the binomial theorem, and the trinomial theorem.

26. *On the pivotal element method for determinants and systems of linear equations*, by Professor M. G. Boyce, Vanderbilt University.

The pivotal element method of solving systems of linear equations, a systematic elimination of unknowns, has the advantages of easy checking and adaptability to machine computation. Its obvious relationship to determinant methods suggests defining determinants by their pivotal element evaluations. Such a definition is shown to define uniquely the value of a determinant, independent of the choice of pivotal elements, and to afford a ready derivation of the elementary properties of determinants.

27. *On a determinant of Sylvester*, by Professor F. A. Lewis, University of Alabama.

If the elements of a determinant V of order n are defined by $v_{rc}=E^{(r-1)(c-1)}$, where $E=e^{2\pi i/n}$, V is reducible if n is odd. Formulae were given for values of the component determinants whose elements involve only cosines and sines respectively.

28. *A formula for tan nx*, by Professor J. A. Ward, University of Georgia.

The formula

$$\tan nx = \left[\sum_{r=0}^{\infty}(-1)^r\binom{n}{2r+1}\tan^{2r+1}x\right]\left[\sum_{r=0}^{\infty}(-1)^r\binom{n}{2r}\tan^{2r}x\right]^{-1}$$

was derived. It was pointed out that the numerator and denominator have only a finite number of terms each when n is a positive integer. Otherwise they are series which converge when $|\tan x|<1$. If $|\tan x|>1$, then $\tan nx$ may be expressed in a similar manner by convergent series in $\cot x$.

29. *A generalized Vandermonde determinant*, by E. B. Shanks, Vanderbilt University.

Let $D_i(x)$ be a matrix of n rows and i columns ($n \geqq i$) such that the element in the pth row and qth column is the qth term of the binomial expansion of $(x+1)^{p-1}$ when q is not greater than p, all other elements being zero. Let S be a determinant of order n whose columns consist of r blocks, where a typical block contains the i_k columns of $D_{i_k}(a_k)$. When $i_k=1$ for all k, S is the Vandermonde determinant. The purpose of this paper was to prove that S is equal to the product of all factors of the type $(a_k-a_j)^{i_k i_j}$, $j<k$.

30. *The radical axis of a pair of conics*, by Lillian G. Perkins, University of South Carolina.

The generalization of the radical axis of a pair of circles was made for a pair of conics. The general case for any two conics was discussed, and then theorems were obtained for some special cases of ellipses, hyperbolas and parabolas.

31. *The use of projection in the proof of certain theorems of triangles*, by Professor W. V. Parker, University of Georgia.

Every triangle is an orthogonal projection of an equilateral triangle. The ellipse of minimum area circumscribing the triangle has its center at the centroid of the triangle and is the orthogonal projection of the circumscribing circle for the equilateral triangle. By using the properties which are invariant under orthogonal projection many theorems for general triangles may be proved by proving them for equilateral triangles only.

H. A. ROBINSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Annual Meeting, New York City, December 30, 1949.

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950

INDIANA, Wabash College, Crawfordsville, April 29, 1950

IOWA, State University of Iowa, Iowa City, April 21–22, 1950

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, Spring, 1950

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Fall, 1949

METROPOLITAN NEW YORK, Spring, 1950

MICHIGAN, March, 1950

MINNESOTA, University of North Dakota, Grand Forks, October 15, 1949

MISSOURI, Spring, 1950

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950

NORTHERN CALIFORNIA, Berkeley, January 28, 1950

OHIO, Denison University, Granville, April 22, 1950

OKLAHOMA, Oklahoma City, October 14, 1949

PACIFIC NORTHWEST, University of Washington, Seattle, June, 1950

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN, University of Denver, April, 1950

SOUTHEASTERN, University of Florida, Gainesville, March, 1950

SOUTHERN CALIFORNIA, Immaculate Heart College, Hollywood, March 11, 1950

SOUTHWESTERN

TEXAS, Abilene, Spring, 1950

UPPER NEW YORK STATE, Syracuse University, Spring, 1950

WISCONSIN, Marquette University, Milwaukee, May, 1950

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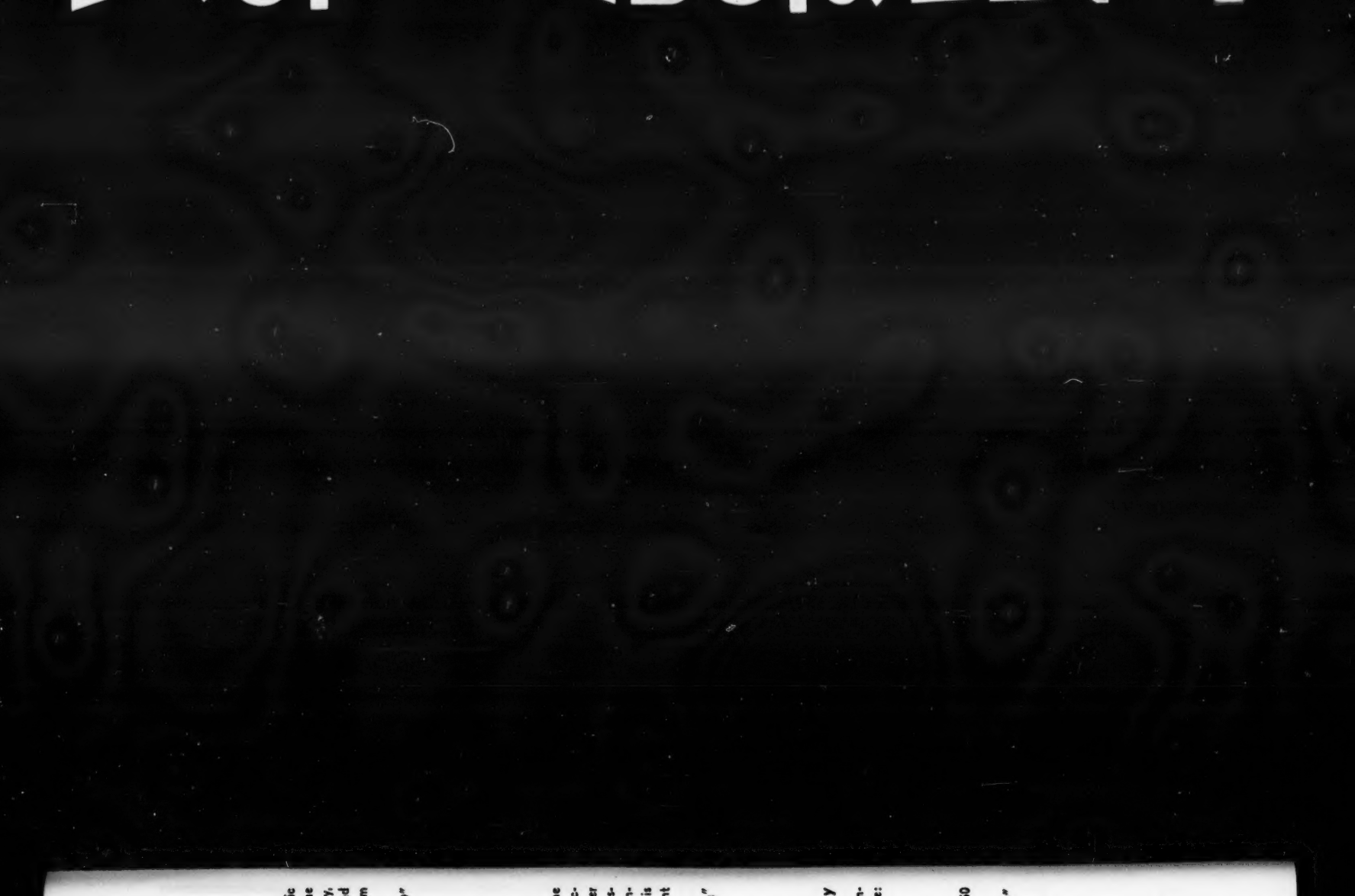

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8

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CONTENTS

A Problem on Arc Tangent Relations . . . . . JOHN TODD 517
An Approximation to the Quotient of Gamma Functions . . . . . J. S. FRAME 529
A Problem in Difference Sets . . . . . MOSHE LOTAN 535
The Rational Canonical Form of a Matrix II . . M. F. SMILEY 542
Mathematical Notes . . . KARL MENGER, VICTOR THÉBAULT 545
Classroom Notes. . . . . L. M. COURT, L. J. BURTON 547
Elementary Problems and Solutions 552
Advanced Problems and Solutions 556
Recent Publications 562
Clubs and Allied Activities 568
News and Notices 573
Mathematical Association of America 578
New Members 578
The March Meeting of the Southern California Section 580
The April Meeting of the Michigan Section 583
The April Meeting of the Louisiana-Mississippi Section . . . 586
The April Meeting of the Texas Section 588
The April Meeting of the Iowa Section 590
Calendar of Future Meetings 594

OCTOBER 1949